

CLASSICALITY FOR SMALL SLOPE OVERCONVERGENT AUTOMORPHIC FORMS ON SOME COMPACT PEL SHIMURA VARIETIES OF TYPE C

CHRISTIAN JOHANSSON

ABSTRACT. We study the rigid cohomology of the ordinary locus in some compact PEL Shimura varieties of type C with values in automorphic local systems and use it to prove a small slope criterion for classicality of overconvergent Hecke eigenforms. This generalises the work of Coleman, and is a first step in an ongoing project to extend the cohomological approach to classicality to higher-dimensional Shimura varieties.

INTRODUCTION

A celebrated theorem of Coleman states that if f is an overconvergent modular form of weight k which is an eigenform for U_p with slope (i.e. the p -adic valuation of the eigenvalue) less than $k - 1$, then f is in fact a (classical) modular form of weight k for the congruence subgroup $\Gamma_1(N) \cap \Gamma_0(p)$. This theorem, usually referred to either as a “classicality theorem” or “control theorem”, generalized a previous result of Hida for ordinary p -adic modular forms and is the key result needed for extending constructions on classical modular forms (such as constructing Galois representations) to overconvergent modular forms of finite slope by p -adic interpolation since it implies that classical forms are dense in Coleman families and on the Coleman-Mazur eigencurve. The links between finite slope overconvergent modular forms and Galois representations which are trianguline at p were investigated by Kisin in [Kis], where it was shown that these Galois representations satisfy the Fontaine-Mazur conjecture.

In attempting to generalize Coleman’s geometric theory for p -adic interpolation of modular forms to other PEL Shimura varieties one quickly runs into two major obstacles; defining families and proving the analogue of the classicality criterion. Both problems seem hard, as Coleman’s methods do not generalize in an obvious way (using methods similar to those of Coleman, Kisin and Lai constructed one-dimensional families of Hilbert modular forms; this has recently been extended to the Siegel-Hilbert case by Mok and Tan [MoTa]). Instead other methods of p -adic interpolation were developed (see e.g. [Buzz], [Che], [Loe], [Eme] and [Urb]), which have been applied with great success to the deformation theory of Galois representations.

Recently there has been much progress on also in the geometric theory, using methods that are very different to Coleman’s; see [AIP] for the construction of families and [PiSt1], [PiSt2] and [Tian] for classicality results. The method for proving classicality originates from work of Kassaei [Kas2], building on previous work by Buzzard and Taylor [BuTa] on the 2-dimensional Artin conjecture, and is in essence a geometric way of analytically continuing the overconvergent form to the whole modular curve (or more generally Shimura variety) of Iwahori level at p . In particular it is entirely different from Coleman’s proof, which is cohomological in nature, and instead requires a very explicit understanding of the geometry of the Shimura variety and the geometry of the U_p -correspondence.

In this paper, we revisit Coleman's original method and generalize it to certain compact PEL Shimura varieties of type C , which are closely related to Hilbert modular varieties. For the exact definitions of objects and results mentioned in this introduction we refer to the main body of the text. To define our Shimura varieties, we start with a quaternion division algebra B over a totally real field F of degree d over \mathbb{Q} . We fix a rational prime p and assume that p is inert in F , that B is split at p , and also split at every real place of F . Such a B then defines a PEL data in a standard way, hence a reductive group G^* over \mathbb{Q} and given an open compact subgroup $K \subseteq G^*(\mathbb{A}^\infty)$ we get an associated Shimura variety. For a special choice of $K = K_1(c, N)$, let us denote the corresponding Shimura variety by X . It has good reduction at p and we may study the ordinary locus $X_{\mathbb{F}_p}^{ord} \subseteq X_{\mathbb{F}_p}$ in characteristic p and its lift X_{rig}^{ord} inside the rigid analytification of the generic fibre X_{rig} of X , and classical and overconvergent automorphic forms on X_{rig} and X_{rig}^{ord} , defined as (overconvergent) sections of the appropriate sheaf (see section 2.1), these carry actions of appropriate Hecke algebras. Our main theorem is the following:

Theorem. 1) (Theorem 34(b)) Assume that f is an overconvergent Hecke eigenform of weight (k_1, \dots, k_d) ($k_i \geq 2$ for all $1 \leq i \leq d$) with U_p -slope less than $\inf \left(k_i - 1, \frac{\sum k_i}{d} - 2 \right)$ (here p has valuation 1). Then its system of Hecke eigenvalues comes from the p -stabilization of a classical form of level K .

2) (Theorem 36(b)) Assume that f is an overconvergent Hecke eigenform of weight $k \geq 2$ with U_p -slope less than $k - 1$. Then its system of Hecke eigenvalues is classical of level $\Gamma_1(N) \cap \Gamma_0(pq_1 \dots q_r)$ (where the $q_i \neq p$ are the primes where B is ramified).

Let us briefly outline the contents of the paper. Section 1 is devoted to setting up the basic definitions of B , G^* and the Shimura varieties involved. We recall their integral models and the algebraic representation theory of G^* , including the BGG resolution of an irreducible representation. In section 2 we define p -adic and overconvergent automorphic forms on X using the automorphic vector bundles of Harris and Milne and define the Hecke operators acting on them. We give two definitions in particular of the U_p -operator and show that they agree. As in the theory for modular curves one of the definitions uses the canonical subgroup and therefore establishes a very direct link to the Frobenius morphism in characteristic p . A key construction in Coleman's proof is that of a sheaf homomorphism

$$\theta = \theta^{k-1} : \omega^{2-k} \rightarrow \omega^k$$

As is no doubt well-known to experts, this is Faltings's BGG complex for the modular curve (and weight k). In section 2.4 we give the analogues on X . In particular, this gives a "theta map"

$$\theta : \bigoplus_i H^0(X_{rig}, W^\dagger(k_1, \dots, 2 - k_i, \dots, k_d)) \longrightarrow H^0(X_{rig}, W^\dagger(k_1, \dots, k_d))$$

for weights with $k_i \geq 2$ for all i ; here $H^0(X_{rig}, W^\dagger(k'_1, \dots, k'_d))$ denotes the spaces of overconvergent automorphic forms of weight (k'_1, \dots, k'_d) .

Section 3 is the main part of the paper. We begin by recalling some notions from rigid cohomology and overconvergent de Rham cohomology, and define certain overconvergent F -isocrystals \mathcal{E}_k on $X_{\mathbb{F}_p}$ that play a key role in the arguments, analogous to the sheaves \mathcal{H}_k defined in section 2 of [Col]. In section 3.1 we prove the main comparison theorem, analogous to Theorem 5.4 of [Col]. It identifies, in particular, the cokernel of θ with the degree d rigid cohomology of \mathcal{E}_k on X^{ord} , via Faltings's BGG complex. Section 3.2 proves the analogue of the crucial but innocent-looking

Lemma 6.2 of *op. cit.*, showing that forms of slope less than $\inf k_i - 1$ are not in image of θ and hence that their system of Hecke eigenvalues occur in the cohomology of \mathcal{E}_k .

So far the arguments have made no essential use of any specific properties of our Shimura varieties; indeed the results and proofs would carry over for example to any compact PEL Shimura variety with nonvanishing Hasse invariant, or any PEL Shimura curve with nonvanishing Hasse invariant. In section 3.3 we use the excision sequence to reduce the understanding of the degree d rigid cohomology of \mathcal{E}_k on X^{ord} to understanding the degree d cohomology on X and the degree $d + 1$ local cohomology on the complement. The former is well understood, using comparison theorems between various cohomology theories, by the classical theory of automorphic forms (Matsushima's formula). For the latter, we use the observation that a nonordinary abelian variety for our moduli problem has no zero-slopes in its Dieudonné module and some results of Kedlaya [Ked2] to prove bounds for the Frobenius-slopes. This is where the assumption that p is inert in F comes into play. The next section then translates these bounds into information about the U_p -operator, using the link between U_p and Frobenius given by the canonical subgroup, and deduces part 1) of our main theorem above. Finally, for completeness, the last section treats the case $F = \mathbb{Q}$ using a (somewhat simplified) version of Coleman's dimension-counting argument, establishing part 2) of the main theorem.

Let us make some remarks regarding our results. First of all, what we prove is that certain systems of Hecke eigenvalues are classical, rather than the stronger fact that the forms themselves are classical. This is the price we pay for working with Hecke modules and the flexibility they offer. If one had some control on the dimension of the Hecke modules we work with (as Coleman has in [Col]) or knew multiplicity one for overconvergent automorphic forms one could recover the classicality of the forms themselves, but these results are not available in our setting (except when $F = \mathbb{Q}$ where the first technique is available to us, see Remark 40). However, for applications to eigenvarieties and Galois representations this weakening is unimportant, as one passes directly to systems of Hecke eigenvalues anyway. As for optimality, the results of the paper are close to what is expected; indeed one would conjecture that an overconvergent eigenform of slope less than $\inf (k_i - 1)$ has a classical system of Hecke eigenvalues. Our theorem proves this, except when the weight is parallel (for $d \geq 2$) where the bound we obtain is $< k_i - 2$ instead. This is slightly better than the bound $< \inf (k_i - d)$ obtained in [PiSt1] in the Hilbert setting. In view of the result of section 3.2, one could refine this conjecture to say that any overconvergent eigenform not in the image of θ is classical (for modular curves this is Corollary 7.2.1 of [Col]). We prove this in our case when $F = \mathbb{Q}$ and obtain a partial result in this direction (Theorem 34(a)) when $F \neq \mathbb{Q}$, of which part 1) of the main theorem above is a Corollary.

Next, let us discuss the possibility of extending the methods to other Shimura varieties. As mentioned above, everything up until section 3.3 generalizes e.g. to the case of compact PEL Shimura varieties with a nonvanishing Hasse invariant (or indeed an affine generalized ordinary locus), however everything after that depends heavily on the specifics of our moduli problem, in particular it does not generalize to the case where p is unramified in F . There should be a different, though considerably more technical, way of completing the proof using the results of Shin [Shin] and a comparison between trace formulas in p -adic (rigid) and ℓ -adic (étale) cohomology, which should allow for a substantial generalization of our results. We are currently working out the details for some unitary Shimura varieties studied by Harris-Taylor [HaTa] and Taylor-Yoshida [TaYo].

Finally, we should also mention that, in work recently announced, Harris, Lan, Taylor and Thorne use overconvergent automorphic forms on certain non-compact unitary Shimura varieties to associate Galois representations to regular algebraic cuspidal automorphic representations of GL_n over a CM or totally real field, removing the “essentially self-dual” assumption required in previous theorems of this kind. In fact, at the heart of their method is the consideration of the very same cohomology groups (in the analogous situation) that are used in this paper and by Coleman. We should also mention that after a revised version of this paper was submitted for publication, we were made aware of ongoing work of Tian and Xiao [TiXi] on classicality for overconvergent Hilbert modular forms. They make a detailed study of the Ekedahl-Oort stratification and obtain very complete results about the structure of the rigid cohomology of the ordinary locus as a Hecke module in order to deduce classicality for small slope overconvergent Hilbert modular forms using techniques similar to those of this paper.

Acknowledgements. The author would like to his PhD supervisor Kevin Buzzard for suggesting this problem and for his constant help and encouragement during every aspect of this project. He would also like to thank his second supervisor Toby Gee for valuable advice during the write-up, as well as Wansu Kim, James Newton, Shu Sasaki and Teruyoshi Yoshida for many helpful discussions relating to this work, and Francesco Baldassarri and Bernard Le Stum for answering questions about rigid and overconvergent de Rham cohomology. The author wishes to thank the EPSRC for supporting him throughout his doctoral studies. Finally, it is a pleasure to thank the Fields Institute, where the final stages of the write-up of this paper was done, for their support and hospitality as well as excellent working conditions.

1. THE GROUPS AND THE SHIMURA VARIETIES

Throughout this article we fix a rational prime p .

1.1. Groups and algebras. Let F be a totally real field of degree d over \mathbb{Q} in which p is inert, with ring of integers \mathcal{O}_F . We let B denote a totally indefinite quaternion algebra over F , which we in addition assume to be split at p and a division algebra, i.e. not equal to $M_{2/F}$. Denote by \mathcal{O}_B a maximal order of B , which will be fixed throughout the paper. The group of invertible elements \mathcal{O}_B^\times is the \mathcal{O}_F -points of an algebraic group, and we denote by G the restriction of scalars of this group to \mathbb{Z} , i.e. for any ring R :

$$G(R) = (\mathcal{O}_B \otimes_{\mathbb{Z}} R)^\times$$

The reduced norm map $\det : \mathcal{O}_B^\times \rightarrow \mathcal{O}_F^\times$ defines a homomorphism of algebraic groups $\det : G \rightarrow \text{Res}_{\mathbb{Z}}^{\mathcal{O}_F} \mathbb{G}_m$. We define an algebraic subgroup $G^* \subseteq G$ by the cartesian diagram

$$\begin{array}{ccc} G_{/\mathbb{Z}}^* & \longrightarrow & G_{/\mathbb{Z}} \\ \downarrow \det & & \downarrow \det \\ \mathbb{G}_{m/\mathbb{Z}} & \longrightarrow & \text{Res}_{\mathbb{Z}}^{\mathcal{O}_F} \mathbb{G}_{m/\mathcal{O}_F} \end{array}$$

where the lower horizontal map is the injection given on R -points by $R^\times \rightarrow (\mathcal{O}_F \otimes_{\mathbb{Z}} R)^\times$, $r \mapsto 1 \otimes r$. Note that the R -points of G^* are

$$G^*(R) = \{g \in (\mathcal{O}_B \otimes_{\mathbb{Z}} R) \mid \det(g) \in R^\times\}$$

Let E be a finite extension of F that splits G and such that there is a prime \mathfrak{p} above p such that $E_{\mathfrak{p}} = F_p$. We fix a Borel subgroup of G over E and by intersecting it with G^* one gets a Borel B^* of G^* . We fix maximal tori T and T^* of G and G^* defined over E . Since B is split at p we note that

$$G^*(\mathbb{Z}_p) = \{g \in \mathrm{GL}_2(\mathcal{O}_{F_p}) \mid \det(g) \in \mathbb{Z}_p^\times\}$$

$$G^*(\mathbb{Q}_p) = \{g \in \mathrm{GL}_2(F_p) \mid \det(g) \in \mathbb{Q}_p^\times\}$$

The \mathbb{C} -points of G^* and T^* may be described as follows (given an enumeration of the infinite places, which we will fix from now on)

$$G^*(\mathbb{C}) = \left\{ (g_i)_i \in \prod_{i=1}^d \mathrm{GL}_2(\mathbb{C}) \mid \det(g_i) = \det(g_j) \forall i \neq j \right\}$$

$$T^*(\mathbb{C}) = \{(g_i)_i \in G^*(\mathbb{C}) \mid g_i \text{ diagonal } \forall i\}$$

The center of \mathcal{O}_B is \mathcal{O}_F , hence the center of G is $\mathrm{Res}_{\mathbb{Z}}^{\mathcal{O}_F} \mathbb{G}_m$ and the center Z^* of G^* is \mathbb{G}_m . We have (with the above description of $G^*(\mathbb{C})$)

$$Z^*(\mathbb{C}) = \{(\lambda I)_i \in G^*(\mathbb{C}) \mid \lambda \in \mathbb{C}^\times\}$$

The derived group of \mathcal{O}_B^\times (as an algebraic group over \mathcal{O}_F) consists of the elements of reduced norm 1. It follows that the derived subgroup of both G and G^* is the kernel of the reduced norm map \det . As it is the same for both G and G^* , we will denote it by G^{der} . We have

$$G^{der}(\mathbb{C}) = \prod_{i=1}^d \mathrm{SL}_2(\mathbb{C})$$

We fix a maximal torus T^{der} of G^{der} over E and make it so that

$$T^{der}(\mathbb{C}) = T^*(\mathbb{C}) \cap G^{der}(\mathbb{C}) = \left\{ \begin{pmatrix} a_i & & \\ & a_i^{-1} & \\ & & 1 \end{pmatrix}_i \in \prod_{i=1}^d \mathrm{SL}_2(\mathbb{C}) \mid a_i \in \mathbb{C}^\times \right\}$$

1.2. Representation theory of G^* . In this section we describe the finite dimensional representation theory of G^* and its weights and central characters. As with any reductive group, its finite dimensional irreducible representations are given by a finite dimensional irreducible representation of its derived group together with a matching central character, where matching means that the representation and the central character must agree on the intersection between the derived group and the center.

Remark 1. The intersection of $G^{der}(\mathbb{C})$ and $Z^*(\mathbb{C})$ is $\{\pm I\} = \left\{ \pm \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}_i \in \prod_{i=1}^d \mathrm{SL}_2(\mathbb{C}) \right\}$, so we need to check compatibility on the element $-I$.

The representations of SL_2 are well known and gives us the following:

Proposition 2. *The irreducible finite dimensional representations of $G^{der}(\mathbb{C})$ are parametrized by d -tuples of non-negative integers (k_1, \dots, k_d) , corresponding to the representation*

$$\bigotimes_{i=1}^d \mathrm{Sym}^{k_i}(\mathrm{Sd}_i)$$

where Sd_i is the representation given by projection $G^{der}(\mathbb{C}) = \prod_{i=1}^d \mathrm{SL}_2(\mathbb{C}) \rightarrow \mathrm{SL}_2(\mathbb{C})$ onto the i -th factor together with the standard (left) representation of $\mathrm{SL}_2(\mathbb{C})$ on \mathbb{C}^2 . All these representations can be defined over any field extension of F that splits B , in particular E . The element $-I$ acts on $\bigotimes_{i=1}^d \mathrm{Sym}^{k_i}(Sd_i)$ by $(-1)^{k_1+\dots+k_d}$.

Since $Z^* \cong \mathbb{G}_m$ we deduce

Corollary 3. *The irreducible finite dimensional representations of $G^*(\mathbb{C})$ are parametrized by $d+1$ -tuples of integers (k_1, \dots, k_d, w) , with $k_i \geq 0$ for all i and $w \equiv \sum k_i \pmod{2}$, and this corresponds to the representation*

$$\left(\bigotimes_{i=1}^d \mathrm{Sym}^{k_i}(Sd_i) \right) \otimes \det^{(w-\sum k_i)/2}$$

Here, similar to before Sd_i is the representation given by projection $G^*(\mathbb{C}) \subseteq \prod_{i=1}^d \mathrm{GL}_2(\mathbb{C}) \rightarrow \mathrm{GL}_2(\mathbb{C})$ onto the i -th factor together with the standard (left) representation of $\mathrm{GL}_2(\mathbb{C})$ on \mathbb{C}^2 , and \det is the reduced norm representation. $\bigotimes_{i=1}^d \mathrm{Sym}^{k_i}(Sd_i)$ corresponds to $(k_1, \dots, k_d, \sum k_i)$ and \det corresponds to $(0, \dots, 0, 2)$. As before, all representations can be defined over any extension of F that splits B , in particular E .

We have a similar description of the characters of T^* and T^{der} :

Proposition 4. *The characters of $T^{der}(\mathbb{C})$ are parametrized by d -uples of integers (k_1, \dots, k_d) and the characters of $T^*(\mathbb{C})$ are parametrized by $d+1$ -uples (k_1, \dots, k_d, w) of integers such that $w \equiv \sum k_i \pmod{2}$. We will denote the corresponding characters by $\chi(k_1, \dots, k_d)$ resp. $\chi(k_1, \dots, k_d, w)$.*

Next we wish to describe a representation which will be important in what follows. This is the representation, defined over \mathbb{Z} , given by the standard left action of $G^*(R) \subseteq (\mathcal{O}_B \otimes_{\mathbb{Z}} R)^\times$ on $\mathcal{O}_B \otimes_{\mathbb{Z}} R$, and we will denote it Sd . Over any extension that the Sd_i are defined over, it splits non-canonically as $Sd = \bigoplus_i (Sd_i \oplus Sd_i)$. The representations we will be working with are the symmetric powers $\mathrm{Sym}^k(Sd)$. We get that

$$\mathrm{Sym}^k(Sd) = \mathrm{Sym}^k \left(\bigoplus (Sd_i \oplus Sd_i) \right) = \bigoplus_{(k_1, \dots, k_d, k'_1, \dots, k'_d)} \bigotimes_i \left(\mathrm{Sym}^{k_i - k'_i}(Sd_i) \otimes \mathrm{Sym}^{k'_i}(Sd_i) \right)$$

where the sum in the furthest right hand side is taken over all $2d$ -uples of non-negative integers $(k_1, \dots, k_d, k'_1, \dots, k'_d)$ such that $\sum k_i = k$ and $0 \leq k'_i \leq k_i$. Moreover, we have

$$\mathrm{Sym}^{k_i - k'_i}(Sd_i) \otimes \mathrm{Sym}^{k'_i}(Sd_i) = \bigoplus_{0 \leq a_i \leq k_i/2} (\mathrm{Sym}^{k_i - 2a_i}(Sd_i) \otimes \det^{a_i})$$

where the a_i are integers. Putting it together we have

$$\mathrm{Sym}^k(Sd) = \bigoplus_{(k_1, \dots, k_d, a_1, \dots, a_d)} \left(\bigotimes_i \mathrm{Sym}^{k_i - 2a_i}(Sd_i) \right) \otimes \det^{\sum a_i}$$

with the k_i and a_i as above. Note that $(\bigotimes_i \mathrm{Sym}^{k_i - 2a_i}(Sd_i)) \otimes \det^{\sum a_i}$ corresponds to $(k_1 - 2a_1, \dots, k_d - 2a_d, k)$, i.e. all these representations have the same central character.

1.3. Shimura varieties defined by G^\star . In this section we briefly recall some well known constructions, see e.g. [Kott], [Mil3] and [Lan]. B carries an involution $b \mapsto b^\star$ of the first kind. Consider the opposite \mathbb{Q} -algebra B^{op} with involution $b \mapsto b^\star$, with the natural left action on B . Pick $\xi \in B$ such that $\xi^\star = -\xi$, and define a B^{op} -involution on B by $(x, y) = Tr_{F/\mathbb{Q}} Tr_{B/F}(x^\star \xi y)$, where $Tr_{B/F}$ is the reduced trace and $Tr_{F/\mathbb{Q}}$ is the field trace. Together with the homomorphism $h : \mathbb{C} \rightarrow End_{B^{op} \otimes_{\mathbb{Q}} \mathbb{R}}(B \otimes_{\mathbb{Q}} \mathbb{R}) = M_2(F \otimes_{\mathbb{Q}} \mathbb{R})$ given by

$$a + bi \mapsto \begin{pmatrix} 1 \otimes a & -1 \otimes b \\ 1 \otimes b & 1 \otimes a \end{pmatrix}$$

This defines a rational PEL-datum of type C, and hence a Shimura datum whose group is G^\star acting on the disconnected Hermitian symmetric domain $(\mathcal{H}^+)^d \sqcup (\mathcal{H}^-)^d$, where \mathcal{H}^+ is the upper and \mathcal{H}^- is the lower half plane. The associated Shimura varieties are moduli spaces for abelian varieties with extra structures and are defined over the reflex field \mathbb{Q} . For a given neat compact open subgroup $K \subseteq G^\star(\mathbb{A}^\infty)$ we denote the corresponding canonical model of the Shimura variety by Sh_K . For primes not dividing the level or the discriminant of F or B , these have good reduction. We will now very briefly recall the construction of these smooth integral models.

We will use Kottwitz's moduli problem ([Kott]) to define our integral models. As a reference see [Lan] (see Definition 1.4.2.4). Fix an open compact subgroup $K = K^p K_p \subseteq G^\star(\mathbb{A}^\infty)$ such that $K_p = G^\star(\mathbb{Z}_p)$, and let $\mathcal{O}_{B^{op},(p)} = \mathcal{O}_{B^{op}} \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$. We define a functor \mathcal{X}_K sending a locally Noetherian $\mathbb{Z}_{(p)}$ -scheme S to the set of equivalence classes of quadruples $(A, \iota, \lambda, \bar{\eta})$ where

- 1) A/S is an abelian scheme of dimension $2d$
- 2) $\iota : \mathcal{O}_{B^{op},(p)} \rightarrow End_S(A) \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$ is a ring homomorphism
- 3) $\lambda : A \rightarrow A^\vee$ is a prime to p -polarization respecting the $\mathcal{O}_{B^{op},(p)}$ -action
- 4) $Lie(A/S)$ with its $\mathcal{O}_{B^{op},(p)}$ -module structure induced by ι satisfies the determinant condition given by $B \otimes_{\mathbb{Q}} \mathbb{R}$ ([Lan] Def. 1.3.4.2)
- 5) $\bar{\eta}$ is a K^p -level structure as defined in Definition 1.3.7.15 of [Lan]

See [Lan] Def. 1.4.2.4 for the exact definitions and definition of equivalence of two quadruples $(A, \iota, \lambda, \bar{\eta})$ and $(A', \iota', \lambda', \bar{\eta}')$. Here we merely note that when $(A, \iota, \lambda, \bar{\eta})$ and $(A', \iota', \lambda', \bar{\eta}')$ are equivalent there is a prime to p -isogeny $f : A \rightarrow A'$ of abelian schemes such that $\lambda = r \circ f^\vee \circ \lambda' \circ f$ for some $r \in \mathbb{Z}_{(p)}^\times$ and $f \circ \iota(b) = \iota'(b) \circ f$ for all $b \in \mathcal{O}_{B^{op},(p)}$ (the actual definition of equivalence requires an additional condition). If K^p is neat ([Lan] Def. 1.4.1.8), this functor is represented by a smooth projective scheme over $\mathbb{Z}_{(p)}$ which we also denote \mathcal{X}_K . The generic fibre \mathcal{X}_K is Sh_K , we will denote it by X .

Remark 5. 1) Assume that $(A, \iota, \lambda, \bar{\eta})$ and $(A', \iota', \lambda', \bar{\eta}')$ are equivalent by the prime to p -isogeny $f : A \rightarrow A'$. Then f induces an isomorphism $A[p^\infty] \rightarrow A'[p^\infty]$ of p -divisible groups which respects the induced actions of $\mathcal{O}_{B^{op},(p)}$ and the polarizations. Thus each equivalence class of $(A, \iota, \lambda, \bar{\eta})$ defines, up to isomorphism, a principally polarized p -divisible group $A[p^\infty]$ of height $4d$ and dimension $2d$ with an action of $\mathcal{O}_{B^{op},p} = \mathcal{O}_{B^{op}} \otimes_{\mathbb{Z}} \mathbb{Z}_p \cong M_2(\mathcal{O}_{F,p})$. By Morita equivalence, this is equivalent to a principally polarized p -divisible group \mathcal{G}_A of height $2d$ and dimension d with an action of $\mathcal{O}_{F,p}$. We further remark that given $(A, \iota, \lambda, \bar{\eta})$ and a finite flat $\mathcal{O}_{B^{op},(p)}$ -stable p -torsion subgroup H of A , dividing out by H gives a well-defined equivalence class $(A/H, \iota', \lambda', \bar{\eta}')$. From now on, we will simply write A for an equivalence class as above.

2) The deformations of \mathcal{G}_A controls the local geometry of the special fibre of \mathcal{X} by Serre-Tate theory. This is identical to the situation in the Hilbert case.

Although we could work with an arbitrary tame level structure K^p for most purposes we will now specify the tame level we will be working with, in analogy with the literature on overconvergent Hilbert modular forms. Let N be a positive integer prime to p . Let $c \in \mathbb{A}_F^\infty$ be a fixed representative of a double coset in $F_+^\times \backslash \mathbb{A}_F^\infty / \hat{\mathcal{O}}_F^\times$, where F_+^\times denotes the totally positive elements of F^\times and $\hat{\mathcal{O}}_F^\times = (\mathcal{O}_F \otimes_{\mathbb{Z}} \hat{\mathbb{Z}})^\times$. Without loss of generality we may assume that c is relatively prime to p . Define

$$K_1^G(N) = \left\{ g \in \mathrm{GL}_2(\mathbb{A}_F^\infty) \mid g \equiv \begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix} \pmod{N} \right\}$$

Finally, we put

$$K_1(c, N) = G^\star(\mathbb{A}_{\mathbb{Q}}^\infty) \cap \begin{pmatrix} c & 0 \\ 0 & 1 \end{pmatrix} K_1^G(N) \begin{pmatrix} c & 0 \\ 0 & 1 \end{pmatrix}^{-1}$$

where the intersection takes place in $\mathrm{GL}_2(\mathbb{A}_F^\infty)$. As c and N are prime to p , $K_1(c, N)_p = G^\star(\mathbb{Z}_p)$. From now on we will fix c and N such that $K_1(c, N)$ is neat, and put $K = K_1(c, N)$, $\mathcal{X} = \mathcal{X}_K$. As $\det(K) = \hat{\mathbb{Z}}^\times$, \mathcal{X} and its special fiber are connected.

Next we will add level structure at p . Define two subgroups $K_0(p)$, $K_0^0(p)$ of $G^\star(\mathbb{Z}_p)$ by

$$K_0(p) = \left\{ g \in G^\star(\mathbb{Z}_p) \mid g \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \pmod{p} \right\}$$

$$K_0^0(p) = \left\{ g \in G^\star(\mathbb{Z}_p) \mid g \equiv \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \pmod{p} \right\}$$

We set $Y = Sh_{K^p K_0(p)}$ and $Z = Sh_{K^p K_0^0(p)}$. Y resp. Z parametrize pairs (A, H) resp. triples (A, H_1, H_2) , where A is a point of X and H , H_1 and H_2 are finite flat $\mathcal{O}_{B^{\mathrm{op}}, (p)}$ -stable subgroups of A of rank p^d which are killed by p and isotropic with respect to λ . Moreover we require that $H_1 \cap H_2 = 0$. The relative representability of these moduli problems over X may be shown by standard methods (they are closed subschemes of Grassmannians). These methods also show that the moduli problem of Y is relatively representable over \mathcal{X} (again as a closed subscheme of a Grassmannian) giving us a projective integral model \mathcal{Y} of Y . We have a projective morphism $\mathcal{Y} \rightarrow \mathcal{X}$ forgetting H and finite etale morphisms $Z \rightrightarrows Y \rightarrow X$ forgetting H_1 resp. H_2 resp. H . Since $\det(K^p K_0(p)) = \det(K^p K_0^0(p)) = \hat{\mathbb{Z}}^\times$, Y and Z are connected.

Let $k = \mathcal{O}_F/(p)$ be the residue field. We will denote the special fibres of \mathcal{X} over \mathbb{F}_p , k resp. $\bar{\mathbb{F}}_p$ by $X_{\mathbb{F}_p}$, X_k resp. $X_{\bar{\mathbb{F}}_p}$, and similarly for \mathcal{Y} .

Remark 6. We will use the notation (A, \dots) as above to denote points of the special and/or generic fibres of our moduli spaces; however we will also use the notation \mathcal{A} , A , $A_{\mathbb{F}_p}$ etc (analogous to \mathcal{X} , X , $X_{\mathbb{F}_p}$ etc.) to denote the (abelian scheme associated to) the universal object over the appropriate moduli space. We hope there will be no confusion arising from this. Occasionally we will use the superscript $^{\mathrm{univ}}$ to distinguish the universal object.

2. AUTOMORPHIC FORMS AND HECKE OPERATORS

2.1. Automorphic vector bundles and automorphic forms. One way to define holomorphic automorphic forms is to use the automorphic vector bundle construction, as described e.g. in [Mil2]. The theorem is the following, and only applies over number fields and therefore applies equally well to X , Y or Z or any other neat level. By abuse of notation, we also let $\chi(k_1, \dots, k_d, w)$ denote the representation of B^\star obtained from $\chi(k_1, \dots, k_d, w)$ by letting the unipotent part of B^\star act trivially.

Theorem 7. *To any finite dimensional representation of B^\star we may functorially associate a vector bundle on X such that equivariant maps between representations go to Hecke-equivariant \mathcal{O}_X -linear maps. To any finite dimensional representation of G^\star we may functorially associate a vector bundle with an integrable connection. These bundles and maps are defined over the same fields as the representations and maps are, and the construction respects direct sums and tensor operations, and the rank of the bundle is the dimension of the representation. We will denote by $W(k_1, \dots, k_d, w)$ the line bundle associated to $\chi(k_1, \dots, k_d, w)$ and by $V(k_1, \dots, k_d, w)$ the vector bundle with connection associated to $\left(\bigotimes_{i=1}^d \text{Sym}^{k_i}(St_i)\right) \otimes \det^{(w - \sum k_i)/2}$. The representation \det goes to the Tate twist $\mathbb{Q}(1)$.*

Definition 8. A (holomorphic) automorphic form of weight (k_1, \dots, k_d, w) and level $K_1(c, N)$ is a global section of $W(k_1, \dots, k_d, w)$ on X (and similarly, changing the level, for Y and Z).

The PEL datum is set up such that the standard representation Sd corresponds to $H_1^{dR}(A/X)$, hence $H_{dR}^1(A/X)$ corresponds to Sd^\vee . Sd^\vee , as a T^\star -representation, is

$$Sd^\vee = (\chi(1, 0, \dots, 0, -1)^{\oplus 2} \oplus \dots \oplus \chi(0, \dots, 0, 1, -1)^{\oplus 2}) \oplus (\chi(-1, 0, \dots, 0, -1)^{\oplus 2} \oplus \dots \oplus \chi(0, \dots, 0, -1, -1)^{\oplus 2})$$

Another bundle that will occur later is Ω_X^d . To start with, Ω_X^1 corresponds to the dual of the adjoint representation of B^\star on $\text{Lie}(G^\star)/\text{Lie}(B^\star) = \chi(2, 0, \dots, 0, 0) \oplus \dots \oplus \chi(0, \dots, 0, 2, 0)$ (note the trivial central character). Therefore $\Omega_X^d = \wedge^d \Omega_X^1$ corresponds to $\chi(2, \dots, 2, 0)$.

Remark 9. 1) Let us briefly explain the relation between this and the perhaps more standard way of defining automorphic forms on X , as in e.g. [Kas1], from which part of this discussion is taken. This will also provide an integral structure to our sheaves of automorphic forms. Fix an identification of $\mathcal{O}_B \otimes_{\mathbb{Z}} \mathbb{Z}_p$ with $M_2(\mathcal{O}_{F_p})$, and consider the two standard orthogonal idempotents e_1 and e_2 in $M_2(\mathcal{O}_{F_p})$. The sheaf $\pi_\star \Omega_{A/X}^1 = e^\star \Omega_{A/X}^1$ injects into $H_{dR}^1(A/X)$ and corresponds to

$$\chi(1, 0, \dots, 0, -1)^{\oplus 2} \oplus \dots \oplus \chi(0, \dots, 0, 1, -1)^{\oplus 2}$$

$\pi_\star \Omega_{A/X}^1$ inherits an action of \mathcal{O}_B and carries a scalar action of \mathbb{Z}_p , hence has an action $\mathcal{O}_B \otimes_{\mathbb{Z}} \mathbb{Z}_p = M_2(\mathcal{O}_{F_p})$. Taking the image of e_2 say (to be consistent with [Kas1]), we obtain a sheaf $\omega = \omega_{A/X}$ which corresponds to

$$\chi(1, 0, \dots, 0, -1) \oplus \dots \oplus \chi(0, \dots, 0, 1, -1)$$

and still carries an action of \mathcal{O}_{F_p} . Decomposing ω with respect to action of \mathcal{O}_{F_p} as in the Hilbert case, we obtain line bundles ω_i corresponding to $\chi(0, \dots, 0, 1, 0, \dots, 0, -1)$ (the 1 in the i -th place), and automorphic forms of weight (k_1, \dots, k_d) are defined as global sections of $\bigotimes \omega_i^{k_i}$. Note that these correspond to our automorphic forms of weight $(k_1, \dots, k_d, -\sum k_i)$, or rather gives an integral structure to this space. We will see when we consider Hecke operators that, the way we are used to thinking about them, automorphic forms of weight (k_1, \dots, k_d) with their usual Hecke action

corresponds to global sections of $\left(\bigotimes \omega_i^{k_i-2}\right) \otimes \Omega_X^d$ (cf. [ChFa] p. 258 for a similar remark in the Siegel case).

2) The central character is only important when considering Hecke operators; the bundles $W(k_1, \dots, k_d, w)$ are isomorphic for fixed (k_1, \dots, k_d) but varying w . Changing w has the effect of scaling Hecke operators, which we will see and use explicitly later. Consequently, we will occasionally just refer to (k_1, \dots, k_d) as the weight and sometimes talk about “an automorphic form of weight (k_1, \dots, k_d) ”, not specifying w , which we will refer to as “the central character”. Sometimes we will include w in the weight. We hope that this will not be confusing.

3) As the $W(k_1, \dots, k_d, w)$ are isomorphic for fixed (k_1, \dots, k_d) and varying w by a canonical isomorphism (see Proposition 24) we may use this isomorphism to define an integral structure on $W(k_1, \dots, k_d, w)$ by transport of structure from $W(k_1, \dots, k_d, -\sum k_i)$.

2.2. Ordinary locus, canonical subgroups and overconvergent automorphic forms. The Hasse invariant is defined as a section of the $(p-1)$ -th tensor power of the Hodge bundle $\wedge^{2d} e^* \Omega_{A/X}^1 \bmod p$ (and can be defined more generally in this fashion for abelian schemes over arbitrary bases), hence a mod p automorphic form of weight $(2p-2, \dots, 2p-2, 2d-2pd)$ on $X_{\mathbb{F}_p}$. The ordinary locus $X_{\mathbb{F}_p}^{ord}$ is the locus where the Hasse invariant does not vanish; its vanishing locus will be denoted $X_{\mathbb{F}_p}^{ss}$ (though it is not the supersingular locus except in low dimensional cases, we hope this will not cause any confusion). $X_{\mathbb{F}_p}^{ord}$ is dense in $X_{\mathbb{F}_p}$ (see [Wed] Thm 1.6.3, note that the reflex field is \mathbb{Q} so E_ν in the notation of [Wed] is \mathbb{Q}_p). The Hodge bundle is ample (see e.g. [Lan] Prop. 7.2.1.1) and hence $X_{\mathbb{F}_p}^{ord}$ is affine (it is the complement of the vanishing locus of a nonzero section of an ample line bundle on a projective variety).

Ultimately we will be interested in rigid-analytic phenomena. When we have a scheme S/\mathbb{Q}_p (or over any extension of valued fields) we will let S_{an} denote its Tate analytification, and whenever we have an scheme \mathcal{S}/\mathbb{Z}_p (or over any extension of valuation rings) we will let S_{rig} denote the Raynaud generic fibre of the completion of \mathcal{S} along its special fibre. S_{rig} carries a specialization map $sp : S_{rig} \rightarrow S_{\mathbb{F}_p}$. When S is the generic fiber of \mathcal{S} there is always an open immersion $S_{rig} \rightarrow S_{an}$ which is an isomorphism when \mathcal{S} is proper. These notions apply to X , Y and Z and their integral models when they exist. Inside $X_{an} = X_{rig}$, with respect to $sp : X_{rig} \rightarrow X_{\mathbb{F}_p}$, we define $X_{rig}^{ord} = sp^{-1}(X_{\mathbb{F}_p}^{ord})$ and $X_{rig}^{ss} = sp^{-1}(X_{\mathbb{F}_p}^{ss})$, the ordinary locus resp. non-ordinary locus in X_{rig} .

Let us briefly recall the construction of the canonical subgroup. The forgetful map $Y_{\mathbb{F}_p} \rightarrow X_{\mathbb{F}_p}$, $(A, H) \mapsto A$, has a partial section $X_{\mathbb{F}_p}^{ord} \rightarrow Y_{\mathbb{F}_p}$, $A \mapsto (A, C_A)$ where C_A is the kernel of the relative Frobenius morphism of A , which is induced by the family $(A_{\mathbb{F}_p}^{univ}, C_{A_{\mathbb{F}_p}^{univ}}) \rightarrow X_{\mathbb{F}_p}^{ord}$. As the Cartier dual of $C_{A^{univ}}/X_{\mathbb{F}_p}^{ord}$ is etale, $(A_{\mathbb{F}_p}^{univ}, C_{A_{\mathbb{F}_p}^{univ}}) \rightarrow X_{\mathbb{F}_p}^{ord}$ lifts uniquely to a family $(A_{\mathbb{Z}/p^n\mathbb{Z}}^{univ}, C_{A_{\mathbb{Z}/p^n\mathbb{Z}}^{univ}}) \rightarrow X_{\mathbb{Z}/p^n\mathbb{Z}}^{ord}$ for any n , which defines compatible system of lifts $X_{\mathbb{Z}/p^n\mathbb{Z}}^{ord} \rightarrow Y_{\mathbb{Z}/p^n\mathbb{Z}}$ of $X_{\mathbb{F}_p}^{ord} \rightarrow Y_{\mathbb{F}_p}$. Taking the limit gives us $\hat{X}^{ord} \rightarrow \hat{Y}$ (where the hat denotes formal completion along the special fibre), and taking generic fibres we obtain a partial section $X_{rig}^{ord} \rightarrow Y_{rig}$, $A \mapsto (A, C_A)$, where C_A is now called the canonical subgroup. The image of this morphism will be denoted Y_{rig}^{ord} .

Remark 10. Y_{rig}^{ord} is not the full ordinary locus in Y , rather it is the so-called ordinary-multiplicative locus. There is also the ordinary-etale locus inside Y , which we will denote (somewhat ad hoc) by

Y_{ord} later; it will only be used once in this paper in an auxiliary role in the construction of the U_p operator.

Let us for completeness give a quick proof that the canonical subgroup overconverges. The proof is an application of a theorem of Berthelot (see the introduction of [KiLa], who attribute this application to de Jong). Before we state it we recall the definition of a frame from [LeSt1] (Def. 3.1.5).

Definition 11. Let K be a complete valued field, let \mathcal{V} be its valuation ring, and k its residue field. A $(K-)$ frame is a diagram

$$S \hookrightarrow T \hookrightarrow P$$

consisting of an open immersion of k -schemes $S \hookrightarrow T$ and a closed immersion of the k -scheme T into a formal \mathcal{V} -scheme P .

We will also write frames as $S \subseteq T \subseteq P$. Frames will be important later when we consider rigid cohomology. Morphisms of frames are simply commutative diagrams

$$\begin{array}{ccccc} S^\circ & \longrightarrow & T^\circ & \longrightarrow & P \\ \downarrow f & & \downarrow g & & \downarrow u \\ S'^\circ & \longrightarrow & T'^\circ & \longrightarrow & P' \end{array}$$

where f and g are morphisms of k -schemes and u is a morphism of formal \mathcal{V} -schemes ([LeSt1] Def 3.1.6). The morphism is said to be quasi-compact if u is quasi-compact [LeSt1] Def 3.2.1), and etale (resp. smooth) if u is etale (resp. smooth) in a neighbourhood of S (inside P) ([LeSt1] Def 3.3.5). The morphism is said to be proper if g is proper ([LeSt1] Def 3.3.10). Given a morphism of frames as above, u induces a morphism $u_K : P_{rig} \rightarrow P'_{rig}$ of rigid analytic varieties which maps $]S[_P$ into $]S'[_{P'}$. We may now state the result referred to above:

Theorem 12. (Berthelot, see [LeSt1] Thm 3.4.12) Let

$$\begin{array}{ccccc} S^\circ & \longrightarrow & T^\circ & \longrightarrow & P \\ \downarrow id & & \downarrow g & & \downarrow u \\ S^\circ & \longrightarrow & T'^\circ & \longrightarrow & P' \end{array}$$

be a proper etale quasi-compact morphism of frames (where id denotes the identity morphism). Then u_K induces an isomorphism between a strict neighbourhood V' of $]S[_{P'}$ in $]T[_P$ and a strict neighbourhood V of $]S[_{P'}$ in $]T'[_{P'}$.

Corollary 13. The canonical subgroup overconverges, i.e. the partial section $X_{rig}^{ord} \rightarrow Y_{rig}$ extends to a partial section $V \rightarrow Y_{rig}$, where V is a strict neighbourhood of X_{rig}^{ord} inside X_{rig} .

Proof. Consider the morphism of frames

$$\begin{array}{ccccc} X_{\mathbb{F}_p}^{ord} & \hookrightarrow & Y_{\mathbb{F}_p} & \hookrightarrow & \hat{\mathcal{Y}} \\ \downarrow id & & \downarrow g & & \downarrow u \\ X_{\mathbb{F}_p}^{ord} & \hookrightarrow & X_{\mathbb{F}_p} & \hookrightarrow & \hat{\mathcal{X}} \end{array}$$

where the open immersion $X_{\mathbb{F}_p}^{ord} \rightarrow Y_{\mathbb{F}_p}$ is the partial section defined above (kernel of Frobenius) and g and u are the forgetful maps. This morphism of frames is proper and quasi-compact, and since the image of $X_{\mathbb{F}_p}^{ord}$ is open inside \hat{Y} and u restricts to an isomorphism on this image it is also étale. Thus by the proposition we may find strict neighbourhoods V' of $]X_{\mathbb{F}_p}^{ord}[$ inside Y_{rig} and V of X_{rig}^{ord} inside X_{rig} such that u_K restricts to an isomorphism $V' \rightarrow V$. The desired extension is then the inverse $V \rightarrow V' \subseteq Y_{rig}$. \square

Quantitative aspects of the overconvergence of the canonical subgroup are interesting and important for applications to analytic continuation problems. They were first studied by Katz in the case of modular curves, and have since been studied by many authors in a variety of situations, see e.g. [GoKa] for a careful study for Hilbert modular varieties. As the local geometry of X , Y and the morphism $Y \rightarrow X$ is identical to the that of the analogous objects in the Hilbert case (via Serre-Tate theory, as noted above), the author believes (though has not checked the details) that the analogues of the results of e.g. [GoKa] should carry over to our situation. This would, for example, imply that U_p (defined in the next section) is completely continuous (cf. [KiLa] Cor. 4.3.6).

Next we will define p -adic and overconvergent automorphic forms. We will abuse notation and use $W(k_1, \dots, k_d, w)$ etc. to denote the analytification of those sheaves on X_{rig} etc.

Definition 14. A p -adic automorphic form of weight (k_1, \dots, k_d, w) is an element of $H^0(X_{rig}^{ord}, W(k_1, \dots, k_d, w))$. An overconvergent automorphic form of weight (k_1, \dots, k_d, w) is an element of

$$H^{0,\dagger}(X_{rig}^{ord}, W(k_1, \dots, k_d, w)) = \varinjlim H^0(V, W(k_1, \dots, k_d, w))$$

where the direct limit is taken over any cofinal set of strict neighbourhoods of X_{rig}^{ord} in X_{rig} . Note that by restriction we have an inclusion $H^{0,\dagger}(X_{rig}^{ord}, W(k_1, \dots, k_d, w)) \subseteq H^0(X_{rig}^{ord}, W(k_1, \dots, k_d, w))$.

2.3. Hecke operators and U_p . We define Hecke operators for our Shimura varieties as in [Kott] §6. For us a special role is played by the Hecke operator U_p , defined adelically on Y by the double coset $K^p K_0^0(p) \begin{pmatrix} p & \\ & 1 \end{pmatrix} K^p K_0^0(p)$, or moduli theoretically by the correspondence

$$(p_1, p_2) : Z \rightarrow Y \times Y$$

where (p_1, p_2) are the two maps given by

$$p_1(A, H_1, H_2) = (A/H_2, A[p]/H_2)$$

$$p_2(A, H_1, H_2) = (A, H_1)$$

One also has the diamond operators $\langle d \rangle : X \rightarrow X$ for $d \in \mathbb{Z}$ with d suitably coprime to K (we will only need the case $d = p$) defined by $\langle d \rangle(A) = A/A[d]$. Note that A and $A/A[d]$ are isomorphic as abelian varieties.

From now on, in this section only, we will only work in the rigid analytic setting and therefore drop the “*rig*” from the notation in order to ease it. We wish to define operators on p -adic and overconvergent automorphic forms and so want to know that the U_p -correspondence restricts to Y^{ord} . Let $Z^{ord} = p_2^{-1}(Y^{ord})$.

Lemma 15. $p_1(Z^{ord}) \subseteq Y^{ord}$

Proof. Let $(A, H_1, H_2) \in Z^{ord}$. By definition $(A, H_1) \in Y^{ord}$, so the reduction of H_1 modulo p is a multiplicative group scheme, and hence H_2 reduces to an etale group scheme. Therefore the reduction of $A[p]/H_2$ is multiplicative, and so $p_1(A, H_1, H_2) = (A/H_2, A[p]/H_2) \in Y^{ord}$. \square

We may therefore restrict to get a correspondence

$$(p_1, p_2) : Z^{ord} \rightarrow Y^{ord} \times Y^{ord}$$

Using the isomorphism $X^{ord} \cong Y^{ord}$ we may view this as a correspondence on X^{ord} , and we may simplify Z^{ord} by noting that the forgetful map $Z \rightarrow Y$ given by $(A, H_1, H_2) \mapsto (A, H_2)$ identifies Z^{ord} with the ordinary-etale locus $Y_{ord} = \{(A, H) \mid A \in X^{ord}, H \neq C_A\}$ in Y , so we get a U_p -correspondence

$$(p_1, p_2) : Y_{ord} \rightarrow X^{ord} \times X^{ord}$$

with

$$p_1(A, H) = A/H$$

$$p_2(A, H) = A$$

Next we wish to define another U_p -correspondence, call it \tilde{U}_p , which will turn out to be isomorphic to U_p . We have a map $Fr : X^{ord} \rightarrow X^{ord}$ given by $Fr(A) = A/C_A$. We denote it Fr because it is a lift of the relative Frobenius in the sense that

$$\begin{array}{ccc} X^{ord} & \xrightarrow{Fr} & X^{ord} \\ \downarrow sp & & \downarrow sp \\ X_{\mathbb{F}_p}^{ord} & \xrightarrow{Fr} & X_{\mathbb{F}_p}^{ord} \end{array}$$

commutes. This will be important when we consider rigid cohomology later. We define \tilde{U}_p as the correspondence

$$(q_1, q_2) : X^{ord} \rightarrow X^{ord} \times X^{ord}$$

where

$$q_1 = id$$

$$q_2 = \langle p \rangle^{-1} Fr$$

Lemma 16. Define two morphisms $\alpha : X^{ord} \rightarrow Y_{ord}$ and $\beta : Y_{ord} \rightarrow X^{ord}$ by

$$\alpha(A) = (A/C_A, A[p]/C_A)$$

$$\beta(A, H) = A/H$$

Furthermore, define an automorphism $\langle p \rangle_Y : Y_{ord} \rightarrow Y_{ord}$ by

$$\langle p \rangle_Y(A, H) = \left(\langle p \rangle(A), \frac{\{a \in A \mid pa \in H\}}{A[p]} \right)$$

Then $\beta\alpha = \langle p \rangle$ and $\alpha\beta = \langle p \rangle_Y$, so β defines an isomorphism $Y_{ord} \cong X^{ord}$.

Proof. We have (equalities as points in the moduli spaces)

$$\beta\alpha(A) = \beta(A/C_A, A[p]/C_A) = \frac{A/C_A}{A[p]/C_A} = \frac{A}{A[p]} = \langle p \rangle A$$

and

$$\alpha\beta(A, H) = \alpha(A/H) = \left(\frac{A/H}{A[p]/H}, \frac{\{a \in A \mid pa \in H\}/H}{A[p]/H} \right) = \langle p \rangle_Y(A, H)$$

where the last equality comes from noting that $A[p]/H$ is the canonical subgroup in A/H (since H non-canonical) and that the map $A/A[p] \rightarrow A$ induced by the p -power map on A sends $\frac{\{a \in A \mid pa \in H\}}{A[p]}$ to H . \square

Finally we may prove

Proposition 17. $U_p \cong \tilde{U}_p$

Proof. By the lemma we know that $X^{ord} \cong Y_{ord}$ via β , so it suffices to prove that $q_1\beta = p_1$ and $q_2\beta = p_2$. Now

$$q_1\beta(A, H) = q_1(A/H) = A/H = p_1(A, H)$$

and

$$q_2\beta(A, H) = q_2(A/H) = \langle p \rangle^{-1} \left(\frac{A/H}{A[p]/H} \right) = \langle p \rangle^{-1} \left(\frac{A}{A[p]} \right) = A = p_2(A, H)$$

\square

We may therefore denote both correspondences by U_p . The description in terms of Fr will prove useful in order to study the slopes of U_p .

It remains to extend U_p to (small) strict neighbourhoods of X^{ord} . This can be done both from the more classical point of view, see [Pill] Prop. 4.8.5, or by overconvergence of the canonical subgroup as proved above. The correspondences hence induce operators on spaces of overconvergent automorphic forms.

Remark 18. 1) The Hecke correspondences away from p preserve the ordinary locus. Hence, again using Prop. 4.8.5 of [Pill], these correspondences overconverge and define operators on overconvergent automorphic forms.

2) To properly let a correspondence $s = (s_1, s_2)$ act on automorphic forms of weight (k_1, \dots, k_d, w) one needs also to specify an isomorphism $s_1^*W(k_1, \dots, k_d, w) \cong s_2^*W(k_1, \dots, k_d, w)$. This is done in general by the theory of automorphic vector bundles. To study p -divisibility of U_p , it is preferable though to have some moduli-theoretic interpretation. It suffices to give such an isomorphism for $\pi_{univ, \star} \Omega_{A^{univ}/X}^1$ respecting the action of $M_2(\mathcal{O}_{F_p})$, as all sheaves of automorphic forms are constructed from this data, and so we may describe automorphic forms as “functions” a la Katz defined on “points” (A, ω) with $\omega \in H^0(A, \Omega_A^1)$. Thus, in order to describe the action of U_p on automorphic forms we need to, given (A, ω) and $B = A/H \in U_p(A)$, functorially associate some $\omega' \in H^0(A/H, \Omega_{A/H}^1)$. This is done by inverting the pullback of differentials along the isogeny $A \rightarrow A/H$.

For our second description of U_p we may first of all ignore $\langle p \rangle^{-1}$, as it only changes the level structure away from p . The natural map involved is then (a priori) the isogeny $B \rightarrow B/C_B = A$ and it would seem natural to use pullback of differentials along this isogeny. These definitions do

not agree however, as the composition $B \rightarrow B/C_B = A \rightarrow A/H = B$ is multiplication by p which induces multiplication by p on differentials, so the two definitions disagree by a factor of p . As is standard, we choose the first definition, and modify the second by the appropriate factor of p . This corresponds geometrically to, rather than using $B \rightarrow A$, using its “dual” $A \rightarrow B$ (defined such that the composition both ways are multiplication by p , and related to the dual isogeny via our polarizations). More explicitly, one has $\tilde{U}_p = p^{-\sum k_i} U_p$ on $H^{0,\dagger}(X_{rig}^{ord}, W(k_1, \dots, k_d, -\sum k_i))$ at first, and then scale so that $\tilde{U}_p = U_p$. Note that whereas the theory of automorphic vector bundles gives definitions of Hecke operators for all weights (k_1, \dots, k_d, w) , we make this moduli-theoretic definition a priori only for weights of the form $(k_1, \dots, k_d, -\sum k_i)$. For general central characters we scale appropriately to match the theory of automorphic vector bundles, cf. Proposition 24.

For the rest of the article we will let \mathcal{H}_K denote the full Hecke algebra of $G^*(\mathbb{A}^\infty)$ with respect to the level K , and let \mathcal{H}_K^p denote the full Hecke algebra of $G^*(\mathbb{A}^{p,\infty})$. Later on when we consider eigenforms we will fix a commutative subalgebra $\mathcal{H}^p \subseteq \mathcal{H}_K^p$ (which is assumed to be full for primes $\ell \neq p$ for which B is split and K^p is maximal) and work with the (commutative) subalgebra $\mathcal{H} = \mathcal{H}^p[U_p, \langle p \rangle] \subseteq \mathcal{H}_K$.

For future use we will define two other correspondences at p . The first is the Frobenius correspondence (or really morphism)

$$Fr : X^{ord} \rightarrow X^{ord} \times X^{ord}$$

with $Fr_1 = Fr$ and $Fr_2 = id$. The second is T_p :

$$T_p : Y \rightarrow X \times X$$

defined by $(T_p)_1(A, H) = A/H$, $(T_p)_2(A, H) = A$. The analytification of T_p preserves the ordinary locus (as ordinarity is preserved by isogenies) and hence we may restrict, obtaining a correspondence

$$T_p : Y^{ord} \sqcup Y_{ord} \rightarrow X^{ord} \times X^{ord}$$

As above both of these correspondences overconverge. Given $A \in X^{ord}$ and $\omega \in H^0(A, \Omega_A^1)$, we have $Fr(A) = A/C_A$ and define a differential $\omega' \in H^0(A/C_A, \Omega_{A/C_A}^1)$ by inverse pullback along $A \rightarrow A/C_A$. This makes Fr act on automorphic forms by Remark 18. For T_p the same discussion as for U_p in Remark 18 applies to give the action on automorphic forms. We remark that, as correspondences, $T_p = U_p + Fr$ (see [Lau1] §1.6 for the definition of addition of correspondences) and with the conventions above T_p and $U_p + Fr$ also induce the same actions on automorphic vector bundles.

2.4. BGG complexes for G^* . We wish to compute the BGG complex of the representation $Sym^{k-2d}(Sd)$, for $k \geq 2d$. For BGG complexes see [BGG] for the original paper and [Hum] for a recent detailed study. For our purpose, the theorem specialized to our situation is the following (the passage from semisimple to reductive Lie algebras merely consists of adding a central character):

Theorem 19. (*BGG resolution*) *If V is the irreducible representation of the reductive Lie algebra $\mathfrak{g}^* = Lie(G^*(\mathbb{C}))$ of dominant weight $\lambda = (k_1, \dots, k_d, w)$, then we have a resolution*

$$0 \rightarrow C_d^V \rightarrow \dots \rightarrow C_0^V \rightarrow V \rightarrow 0$$

with $C_r^V = \bigoplus_{w \in W(r)} \chi(w(\lambda + \rho) - \rho)$. The chain complex C_\bullet^V is a quasi-isomorphic direct summand of the bar resolution D_\bullet^V defined by $D_r^V = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \wedge^r(\mathfrak{g}^/\mathfrak{b}^*) \otimes_{\mathbb{C}} V$, with $\mathfrak{b}^* = Lie(B^*(\mathbb{C}))$.*

Here $W^{(r)}$ denotes the elements in the Weyl group of length r . The Weyl group of $G^*(\mathbb{C})$ is the same as that for its derived group, hence isomorphic to $\{\pm 1\}^d$, and an element $(\epsilon_1, \dots, \epsilon_d)$ acts on a weight (k_1, \dots, k_d, w) by $(\epsilon_1, \dots, \epsilon_d) \cdot (k_1, \dots, k_d, w) = (\epsilon_1 k_1, \dots, \epsilon_d k_d, w)$, and the length of $(\epsilon_1, \dots, \epsilon_d)$ is $\#\{i \mid \epsilon_i = -1\}$. ρ denotes half the sum of the positive roots, which in our case is $(1, \dots, 1, 0)$. The theorem assumes V irreducible; we may treat arbitrary semisimple representations by decomposing and taking direct sums (of course this decomposition may not be unique in general).

Recall from above that

$$Sym^{k-2d}(Sd) = \bigoplus_{(k_1, \dots, k_d, a_1, \dots, a_d)} \left(\bigotimes_i Sym^{(k_i-2)-2a_i}(Sd_i) \right) \otimes \det^{\sum a_i}$$

with the a_i and k_i are integers such that $\sum k_i = k$, $k_i \geq 2$ and $0 \leq a_i \leq (k_i - 2)/2$, with $(\bigotimes_i Sym^{(k_i-2)-2a_i}(Sd_i)) \otimes \det^{\sum a_i}$ irreducible of dominant weight $(k_1 - 2 - 2a_1, \dots, k_d - 2 - 2a_d, k - 2d)$. The BGG complex of $(\bigotimes_i Sym^{(k_i-2)-2a_i}(Sd_i)) \otimes \det^{\sum a_i}$ therefore has r -th term

$$\bigoplus_{(\epsilon_1, \dots, \epsilon_d)} \chi(\epsilon_1(k_1 - 1 - 2a_1) - 1, \dots, \epsilon_d(k_d - 1 - 2a_d) - 1, k - 2d)$$

where the direct sum is taken over all $(\epsilon_1, \dots, \epsilon_d) \in (W^{(r)})^d$. Note that $\epsilon_i(k_i - 1 - 2a_i) - 1$ is $k_i - 2 - 2a_i$ if $\epsilon_i = 1$ and $-k_i + 2a_i$ if $\epsilon_i = -1$. Putting it together, $Sym^{k-2d}(Sd)$ has BGG complex with r -th term

$$\bigoplus_{(k_1, \dots, k_d, a_1, \dots, a_d)} \bigoplus_{(\epsilon_1, \dots, \epsilon_d)} \chi(\epsilon_1(k_1 - 1 - 2a_1) - 1, \dots, \epsilon_d(k_d - 1 - 2a_d) - 1, k - 2d)$$

2.5. Dual BGG complexes for X . The automorphic vector bundle construction produces from the BGG complex of an irreducible representation V a complex of vector bundles and differential operators which is a quasi-isomorphic direct summand of the de Rham complex of the vector bundle with connection associated to V (see e.g. [Fal], [ChFa] or [LaPo]). Specialized to our situation, the theorem is:

Theorem 20. ([Fal] Thm 3, [ChFa]) *We have, associated to the irreducible representation of dominant weight $\lambda = (k_1, \dots, k_d, w)$, over $\overline{\mathbb{Q}}$, a complex*

$$0 \rightarrow \mathcal{K}_\lambda^0 \rightarrow \dots \rightarrow \mathcal{K}_\lambda^d \rightarrow 0$$

called the dual BGG complex, with $\mathcal{K}_\lambda^r = \bigoplus_{w \in W^{(r)}} W(w(\lambda + \rho) - \rho)^\vee$ on X where the maps are Hecke-equivariant differential operators, which is a quasi-isomorphic direct summand of the de Rham complex $V(\lambda)^\vee \otimes_{\mathcal{O}_X} \Omega_X^\bullet$ of $V(\lambda)^\vee$.

Here, as earlier and as will be the case in the rest of the article, $V(\lambda) = V(k_1, \dots, k_d, w)$ denotes the vector bundle with connection associated to $(\bigotimes_{i=1}^d Sym^{k_i}(Sd_i)) \otimes \det^{(w - \sum k_i)/2}$. As in the previous section, we may of course consider arbitrary semisimple representations by decomposing and taking direct sums. Thus we get a BGG complex of $Sym^{k-2d}(H_{dR}^1(A/X))$ (which is associated to $Sym^{k-2d}(Sd)^\vee$), with r -th term

$$(2.1) \quad \bigoplus_{(k_1, \dots, k_d, a_1, \dots, a_d)} \bigoplus_{(\epsilon_1, \dots, \epsilon_d)} W(-\epsilon_1(k_1 - 1 - 2a_1) + 1, \dots, -\epsilon_d(k_d - 1 - 2a_d) + 1, -k + 2d)$$

which is a direct summand of the de Rham complex of $Sym^{k-2d}(H_{dR}^1(A/X))$.

3. RIGID AND OVERCONVERGENT DE RHAM COHOMOLOGY

As references for rigid cohomology we will mainly use [LeSt1], but see also (for example) the papers [Ked1], [Ked2] for a slightly different and perhaps more concrete perspective, or the paper [LeSt2] for a site-theoretic framework paralleling that of crystalline cohomology. We are ultimately interested in the rigid cohomology groups of $X_{\mathbb{F}_p}^{ord}$ (and the overconvergent de Rham cohomology groups of X_{rig}^{ord}) with values in certain overconvergent F -isocrystals (or overconvergent differential modules), considered as Hecke modules and as F -isocrystals. To analyze these we introduce the frames

$$\begin{aligned} A_{\mathbb{F}_p} &= A_{\mathbb{F}_p} \subseteq \hat{A} \\ X_{\mathbb{F}_p}^{ord} &\subseteq X_{\mathbb{F}_p} \subseteq \hat{\mathcal{X}} \\ X_{\mathbb{F}_p} &= X_{\mathbb{F}_p} \subseteq \hat{\mathcal{X}} \\ X_{\mathbb{F}_p}^{ss} &\subseteq X_{\mathbb{F}_p} \subseteq \hat{\mathcal{X}} \end{aligned}$$

Note that there is a map of frames from the first frame above to the third coming from the map $\mathcal{A} \rightarrow \mathcal{X}$. We may use these frames to interpret overconvergent isocrystals (and rigid cohomology) on $X_{\mathbb{F}_p}^{ord}$, $X_{\mathbb{F}_p}$ resp. $X_{\mathbb{F}_p}^{ss}$ as overconvergent (on X_{rig}^{ord} , X_{rig} resp. X_{rig}^{ss}) differential modules (and de Rham cohomology) on X_{rig} , since $X_{\mathbb{F}_p}$ and $A_{\mathbb{F}_p}$ are proper over \mathbb{F}_p and $\hat{\mathcal{X}}$ resp. \hat{A} is smooth in a neighbourhood of $X_{\mathbb{F}_p}$ resp. $A_{\mathbb{F}_p}$ (since they are smooth; this is Cor. 8.1.9 and Prop. 7.2.13 of [LeSt1]). We may also use lifts of Frobenius to calculate Frobenius actions (see [LeSt1] §8.3). It should be noted that functoriality is not as rigid as frames look like; given a frame $X \subseteq Y \subseteq P$ one does not need to lift morphisms to P , it is sufficient to lift them to a strict neighbourhood of $|X|_P$ (tube of X inside P), see [LeSt1] Prop. 8.1.6 (see also [LeSt2], where this observation is built into the foundations).

Consider the universal abelian variety $A_{\mathbb{F}_p} \rightarrow X_{\mathbb{F}_p}$. The relative rigid cohomology $H_{rig}^1(A_{\mathbb{F}_p}/X_{\mathbb{F}_p})$ is a convergent F -isocrystal on X and its fibres over points $x \in X$ are $H_{rig}^1(A_{\mathbb{F}_p, x})$, which is the contravariant Dieudonné module of $A_{\mathbb{F}_p, x}$ with its Frobenius action (see e.g. [Tzu] Thm 4.1.4 for the relevant base change assertion). The morphism from the Frobenius pullback of $H_{rig}^1(A_{\mathbb{F}_p}/X_{\mathbb{F}_p})$ to $H_{rig}^1(A_{\mathbb{F}_p}/X_{\mathbb{F}_p})$ is given by pull back along the relative Frobenius of A/X (this is the induced Frobenius structure on rigid cohomology; see also the remark by the end of §2 of [Col]). By [Tzu] Thm 4.1.4 again, the restrictions of $H_{rig}^1(A_{\mathbb{F}_p}/X_{\mathbb{F}_p})$ to $X_{\mathbb{F}_p}^{ord}$ resp. $X_{\mathbb{F}_p}^{ss}$ are $H_{rig}^1(A_{\mathbb{F}_p}^{ord}/X_{\mathbb{F}_p}^{ord})$ resp. $H_{rig}^1(A_{\mathbb{F}_p}^{ss}/X_{\mathbb{F}_p}^{ss})$ (where $A_{\mathbb{F}_p}^{ord}$ resp. $A_{\mathbb{F}_p}^{ss}$ denotes the restriction of $A_{\mathbb{F}_p}$ to $X_{\mathbb{F}_p}^{ord}$ resp. $X_{\mathbb{F}_p}^{ss}$). The rigid cohomology of these overconvergent F -isocrystals will be our main object of study in this section.

Remark 21. 1) As we are using specific frames to compute rigid cohomology we will think of these rigid cohomology groups and overconvergent de Rham cohomology groups as “the same”; even though we write “ H_{rig} ” from now on we may occasionally want to think of these as overconvergent de Rham cohomology groups. Recall the Hecke algebras \mathcal{H}_K , \mathcal{H}_K^p , \mathcal{H}^p and \mathcal{H} introduced by the end of §2.3. \mathcal{H}_K^p acts as correspondences on \mathcal{X} and both morphisms defining the correspondences are finite étale. Hence we get compatible actions on $X_{\mathbb{F}_p}$ and X_{rig} which preserve $X_{\mathbb{F}_p}^{ord}$ and $X_{\mathbb{F}_p}^{ss}$ (as well as X_{rig}^{ord} and X_{rig}^{ss} , by compatibility). At p we will only consider U_p , T_p , $\langle p \rangle^{\pm 1}$ and Fr . Fr is a Frobenius lift for $X_{\mathbb{F}_p}^{ord}$ and hence gives a concrete way of computing Frobenius actions on the relevant overconvergent F -isocrystals on $X_{\mathbb{F}_p}^{ord}$. Furthermore, Fr, U_p , $\langle p \rangle^{\pm 1}$ and T_p define correspondences with both maps étale on X_{rig} and X_{rig}^{ord} , and will act on the relevant cohomology groups and spaces of automorphic forms on X_{rig} and X_{rig}^{ord} .

2) There is a point of concern of what the natural choice of base field is; when working with automorphic forms it is perhaps \mathbb{C}_p , and \mathbb{Q}_p or a finite extension therefore when working with overconvergent F -isocrystals. We would therefore like to know that our construction commute with the change of base field from a finite extension of \mathbb{Q}_p to \mathbb{C}_p . Rigid cohomology (and coherent cohomology) commutes with a finite extension of base field ([LeSt1] Proposition 8.2.14). However, rigid cohomology is not known in general to commute with change of base field (we are grateful to Le Stum for informing us of this). For us however we may avoid this as follows. First note that for coherent cohomology of complexes on affinoids this is clear, this is just flat base change for modules (there is also no higher coherent cohomology). As our overconvergent de Rham cohomology is just the direct limit of de Rham cohomology taken over a cofinal set of strict neighbourhoods (which we may chose to be affinoid), the assertion follows by exactness of direct limits of modules and the fact that direct limits commute with tensor products. Thus no real problem arises from changing base field. We hope that the reader will find it easy to determine which base field is appropriate throughout this section. The only point that perhaps requires some clarification is that when the base field is not \mathbb{Q}_p , the semilinear Frobenius action on overconvergent F -isocrystals are the semilinearization of the linear Frobenius action on automorphic forms (consider for example the upcoming Theorem 22). When the base field is \mathbb{Q}_p , however, both actions agree, and since slopes for F -isocrystals remain the same after change of base field, the linear Frobenius action and the semilinear Frobenius action on the relevant rigid cohomology groups will have the same slopes. This will be implicitly applied when we compare slopes on rigid cohomology with U_p -slopes in §3.4.

3.1. Relation to overconvergent automorphic forms . Given the cartesian map of frames

$$\left(A_{\mathbb{F}_p} = A_{\mathbb{F}_p} \subseteq \hat{\mathcal{A}} \right) \rightarrow \left(X_{\mathbb{F}_p} = X_{\mathbb{F}_p} \subseteq \hat{\mathcal{X}} \right)$$

and the fact that these both frames realize rigid cohomology, we deduce from the definition of rigid cohomology that the convergent F -isocrystal $H_{rig}^1(A_{\mathbb{F}_p}/X_{\mathbb{F}_p})$ is realized by the de Rham cohomology $R^1(\pi_{rig})_{dR}(\mathcal{O}_{A_{rig}})$ of $\pi_{rig} : A_{rig} \rightarrow X_{rig}$. Since \mathcal{A} and \mathcal{X} are proper we have $R^1(\pi_{rig})_{dR}(\mathcal{O}_{A_{rig}}) = (R^1(\pi_{an})_{dR}(\mathcal{O}_{X_{an}}))$ and by comparison between algebraic and rigid analytic de Rham cohomology (see e.g. [AnBa] Thm. IV.4.1) we have $R^1(\pi_{an})_{dR}(\mathcal{O}_{X_{an}}) = (H_{dR}^1(A/X))_{an}$, hence $R^1(\pi_{rig})_{dR}(\mathcal{O}_{X_{rig}}) = (H_{dR}^1(A/X))_{an}$, and similarly for its symmetric powers.

We will write $V^\dagger(k_1, \dots, k_d, w)$ resp. $W^\dagger(k_1, \dots, k_d, w)$ for $j_{X_{\mathbb{F}_p}^{ord}}^\dagger(V(k_1, \dots, k_d, w)_{an})$ resp. $j_{X_{\mathbb{F}_p}^{ord}}^\dagger(W(k_1, \dots, k_d, w)_{an})$, these are overconvergent sheaves on X_{rig}^{ord} (see [LeSt1] §5.1 for the definition of j^\dagger , it is probably easiest to use his Prop. 5.1.12 as the definition). We may replace $\overline{\mathbb{F}}_p$ by \mathbb{F}_p when the representation is defined over \mathbb{Q} . Applying analytification and $j_{X_k^{ord}}^\dagger$ (both are exact functors) to our dual BGG complexes, we get overconvergent dual BGG complexes $\mathcal{K}_{(k_1, \dots, k_d, w)}^{\dagger, \bullet}$ on X_{rig}^{ord} which are direct summands of corresponding the overconvergent de Rham complexes.

We wish to interpret $H_{rig}^d(X_k^{ord}, V^\dagger(k_1, \dots, k_d, w)^\vee)$ in terms of overconvergent automorphic forms. Since $X_{\mathbb{F}_p}^{ord}$ is affine, X_{rig}^{ord} and its small strict neighbourhoods are quasi-Stein and hence $H^i(X_{rig}, W^\dagger(k_1, \dots, k_d, w)) = 0$ for $i \geq 1$ (coherent cohomology). From this we get the following theorem, which is the analogue of Theorem 5.4 of [Col]:

Theorem 22. *We have*

$$H_{rig}^i(X_k^{ord}, V^\dagger(k_1, \dots, k_d, w)^\vee) = h^i \left(\bigoplus_{(\epsilon_j) \in W(\bullet)} H^0(X_{rig}, W^\dagger(\epsilon_1(k_1 + 1) - 1, \dots, \epsilon_d(k_d + 1) - 1, w)^\vee) \right)$$

Here h^i stands for “ i -th cohomology of the complex”. Thus, in particular, if we denote by $\theta_{(k_1, \dots, k_d, w)}$ the map

$$\bigoplus_{(\epsilon_j) \in W^{(d-1)}} W^\dagger(\epsilon_1(k_1 + 1) - 1, \dots, \epsilon_d(k_d + 1) - 1, w)^\vee \longrightarrow W^\dagger(k_1 + 2, \dots, k_d + 2, -w)$$

and by abuse of notation also the induced map

$$\bigoplus_{(\epsilon_j) \in W^{(d-1)}} H^0(X_{rig}, W^\dagger(\epsilon_1(k_1 + 1) - 1, \dots, \epsilon_d(k_d + 1) - 1, w)^\vee) \longrightarrow H^0(X_{rig}, W^\dagger(k_1 + 2, \dots, k_d + 2, -w))$$

of global sections, then

$$H_{rig}^d(X_{\mathbb{F}_p}^{ord}, V^\dagger(k_1, \dots, k_d, w)^\vee) = \text{Coker } \theta_{(k_1, \dots, k_d, w)}$$

Proof. We have

$$H_{rig}^i(X_{\mathbb{F}_p}^{ord}, V^\dagger(k_1, \dots, k_d, w)^\vee) = H_{dR}^i(X_{rig}, V^\dagger(k_1, \dots, k_d, w)^\vee) = H^i(X_{rig}, \mathcal{K}_{k_1, \dots, k_d, w}^{\dagger, \bullet})$$

where the first equality is by the definition of rigid cohomology (and the fact that $X_{\mathbb{F}_p}^{ord} \subseteq X_{\mathbb{F}_p} \subseteq \hat{\mathcal{X}}$ is a frame that computes rigid cohomology), and the second is by the quasi-isomorphism of the de Rham complex of $V^\dagger(k_1, \dots, k_d, w)^\vee$ and $\mathcal{K}_{k_1, \dots, k_d, w}^{\dagger, \bullet}$. The vanishing $H^i(X_{rig}, W^\dagger(k_1, \dots, k_d, w)) = 0$ then gives the first statement by the hypercohomology spectral sequence. The last statement follows from the first and the definitions. \square

Remark 23. Note that all the previous equalities of cohomology groups are valid as equalities of Hecke modules (cf. Remark 21).

Now look at $H_{rig}^d \left(X_{\mathbb{F}_p}^{ord}, \text{Sym}^{k-2d} (H_{rig}^1(A_{\mathbb{F}_p}/X_{\mathbb{F}_p})) \right)$. Equation 2.1 says that

$$\begin{aligned} & BGG \left(\text{Sym}^{k-2d} (H_{dR}^1(A/X)) \right)^r = \\ &= \bigoplus_{(k_1, \dots, k_d, a_1, \dots, a_d)} \bigoplus_{(\epsilon_1, \dots, \epsilon_d)} W(-\epsilon_1(k_1 - 1 - 2a_1) + 1, \dots, -\epsilon_d(k_d - 1 - 2a_d) + 1, -k + 2d) \end{aligned}$$

Thus

$$H_{rig}^d \left(X_{\mathbb{F}_p}^{ord}, \text{Sym}^{k-2d} (H_{rig}^1(A_{\mathbb{F}_p}/X_{\mathbb{F}_p})) \right) = \bigoplus_{(k_1, \dots, k_d, a_1, \dots, a_d)} \text{Coker } \theta_{(k_1 - 2 - 2a_1, \dots, k_d - 2 - 2a_d, k - 2d)}$$

Fix (k_1, \dots, k_d) , $\sum k_i = k$. One of the summands is $\text{Coker } \theta_{(k_1 - 2, \dots, k_d - 2, k - 2d)}$ which is a quotient of $H^0(X_{rig}, W^\dagger(k_1, \dots, k_d, -k + 2d))$. This is the part of the cohomology we will be interested in.

3.2. Small slope criterion for occurring in the cohomology. We need to determine how to normalize the U_p -operator to achieve optimal p -integrality. This has been done by Hida in [Hid]; the result in our setting is his Theorem 6.8; see also §4. We can reformulate the result as:

Proposition 24. *The U_p -operator is p -integral on $H^0(X_{rig}^{ord}, W(k_1, \dots, k_d, -(\sum k_i) + 2d))$ and hence on $H^0(X_{rig}, W^\dagger(k_1, \dots, k_d, -(\sum k_i) + 2d))$ (in the sense that its eigenvalues are p -integral) and has slope 0-eigenvectors on both these spaces. Moreover, shifting the central character up by 2 scales U_p by p^{-1} .*

Proof. As mentioned before the statement of the proposition, the first part follows from work of Hida and the second part. The proof of the second part is by a standard calculation. Let us outline the argument. First, we prove the analogue statement over the complexes. Let $\Gamma = G^\star(\mathbb{Q}) \cap K$ and let $h = \begin{pmatrix} p & \\ & 1 \end{pmatrix}$. Fix a weight (k_1, \dots, k_d, w) and write $\chi = \chi(k_1, \dots, k_d, w)$. We may interpret automorphic forms of level K and weight (k_1, \dots, k_d, w) over \mathbb{C} as functions

$$f : G^\star(\mathbb{R}) \rightarrow \mathbb{C}$$

satisfying $f(\gamma g) = f(g)$ and $f(gk) = \chi(k)^{-1}f(g)$ for $\gamma \in \Gamma$ and $k \in K_\infty$, or equivalently as functions

$$\phi : G^\star(\mathbb{A}) \rightarrow \mathbb{C}$$

such that $\phi(\gamma g) = \phi(g)$ and $\phi(gk) = \chi(k_\infty)^{-1}\phi(g)$ for $\gamma \in G^\star(\mathbb{Q})$ and $k \in K$ (plus analytic conditions that we will not need and therefore not go into). Given f , the associated ϕ is defined by $\phi(g) = f(g_\infty)$. Note that we may describe local sections of $W(k_1, \dots, k_d, w)$ on $X(\mathbb{C})$ by the same equations, restricting the domain of f to any open U' which is the pullback of some analytic open U under the natural map $G^\star(\mathbb{R}) \rightarrow X(\mathbb{C})$. The adelic operator $U_p = [KhK]$, which in the classical setting becomes $[\Gamma h^{-1}\Gamma]$, acts as

$$(U_p f)(g) = \sum_i f(h^{-1}\gamma_i g)$$

for some (any) set $\gamma_1, \dots, \gamma_r$ of coset representatives of $(\Gamma \cap h\Gamma h^{-1}) \backslash \Gamma$. Now consider changing the weight by a factor of \det , i.e. (k_1, \dots, k_d, w) goes to $(k_1, \dots, k_d, w + 2)$. There is an isomorphism of coherent sheaves

$$\varphi : W(k_1, \dots, k_d, w) \rightarrow W(k_1, \dots, k_d, w + 2)$$

(which is valid over the reflex field) defined on local sections by

$$(\varphi(f))(g) = \det(g)^{-1}f(g)$$

Thus we see that

$$(U_p(\varphi(f)))(g) = \det(h^{-1})^{-1}\det(g)^{-1} \sum f(h\gamma_i g) = p \cdot (\varphi(U_p f))(g)$$

which is the result we wanted. Now as this identity holds analytically over \mathbb{C} , it also holds formally around every \mathbb{C} -point, hence formally around every $\overline{\mathbb{Q}}$ -point, and hence rigid analytically in the ordinary locus by the principle of analytic continuation (the ordinary locus is connected, and contains $\overline{\mathbb{Q}}$ -points). \square

Remark 25. 1) The choice $\phi(g) = f(g_\infty)$ is nonstandard (but seems to the author to be a fairly natural choice). This is what forces U_p to become $[\Gamma h^{-1}\Gamma]$ in the classical setting; it differs from

the usual choice using $\begin{pmatrix} 1 & \\ & p \end{pmatrix}$ by a central factor of $\begin{pmatrix} p & \\ & p \end{pmatrix}$, which reflects the fact that we didn't throw in determinant factors in the equivalence $f \leftrightarrow \phi$.

2) We will also define the action of Fr on $H^0(X_{rig}, W^\dagger(k_1, \dots, k_d, w))$ by using the previously defined action on $H^0(X_{rig}, W^\dagger(k_1, \dots, k_d, -(\sum k_i) + 2d))$ and declaring that shifting the central character up by 2 scales Fr by p^{-1} . This corresponds to the interpretation of the automorphic vector bundle of \det as the Tate twist $\mathbb{Q}_p(1)$.

We may now prove the analogue of Lemma 6.3 of [Col].

Corollary 26. *Let $k_i \geq 2$ for all i . If f is a U_p -eigenform in $H^0(X_{rig}, W^\dagger(k_1, \dots, k_d, -(\sum k_i) + 2d))$ of slope less than $\inf_i(k_i - 1)$, then f is not in the image of θ .*

Proof. Recall that θ is a U_p -equivariant map

$$\bigoplus_i H^0(X_{rig}, W^\dagger(k_1, \dots, 2 - k_i, \dots, k_d, -(\sum k_j) + 2d)) \longrightarrow H^0(X_{rig}, W^\dagger(k_1, \dots, k_d, -(\sum k_i) + 2d))$$

Here the right hand side has the optimal U_p whereas the optimal U_p for weight $(k_1, \dots, 2 - k_i, \dots, k_d)$ occurs with central character $-(2 - k_i + \sum_{i \neq j} k_j) + 2d = -(\sum k_j) + 2d + 2(k_i - 1)$ by the previous Proposition. Thus U_p acting on $H^0(X_{rig}, W^\dagger(k_1, \dots, 2 - k_i, \dots, k_d, -(\sum k_j) + 2d))$ has eigenvalues of slope $\geq k_i - 1$ by the previous Proposition. This proves the Corollary. \square

Thus, again for fixed (k_1, \dots, k_d) , $\sum k_i = k$, $H_{rig}^d(X_{\mathbb{F}_p}^{ord}, Sym^{k-2d}(H_{rig}^1(A_{\mathbb{F}_p}/X_{\mathbb{F}_p})))$ has a sub-Hecke module consisting of the overconvergent automorphic forms of weight $(k_1, \dots, k_d, -k + 2d)$ of U_p -slope $< \inf_i(k_i - 1)$.

3.3. The excision sequence and a slope criterion . The next thing to do is to start analyzing $H_{rig}^d(X_{\mathbb{F}_p}^{ord}, Sym^{k-2d}(H_{rig}^1(A_{\mathbb{F}_p}/X_{\mathbb{F}_p})))$ using the formalism of rigid cohomology. From now on, we will write $\mathcal{E}_k = Sym^{k-2d}(H_{rig}^1(A_{\mathbb{F}_p}/X_{\mathbb{F}_p}))$ and we continue to assume $k_i \geq 2$ for all i . The excision sequence in rigid cohomology gives us a Frobenius-equivariant exact sequence

$$\dots \rightarrow H_{rig}^d(X_{\mathbb{F}_p}, \mathcal{E}_k) \longrightarrow H_{rig}^d(X_{\mathbb{F}_p}^{ord}, \mathcal{E}_k) \longrightarrow H_{X_{\mathbb{F}_p}^{ss}, rig}^{d+1}(X_{\mathbb{F}_p}, \mathcal{E}_k) \rightarrow \dots$$

Here we have some knowledge of the nature of $H_{rig}^d(X_{\mathbb{F}_p}, \mathcal{E}_k)$ as a Hecke module from comparison theorems and “classical” automorphic methods (Matsushima’s formula). The problematic term is the contribution from $H_{X_{\mathbb{F}_p}^{ss}, rig}^{d+1}(X_{\mathbb{F}_p}, \mathcal{E}_k)$. We will deal with it by bounding its slopes. Before we do this we simplify it somewhat as follows:

Proposition 27. $H_{X_{\mathbb{F}_p}^{ss}, rig}^{d+1}(X_{\mathbb{F}_p}, \mathcal{E}_k) \cong H_{rig}^{d-1}(X_{\mathbb{F}_p}^{ss}, \mathcal{E}_k^\vee(d))^\vee$ in a Frobenius and Hecke-equivariant way. Here (d) denotes a Tate twist by d .

Proof. This is just Poincare duality, see [Ked1] Thm 1.2.3 and also [Ked2] §2.1 or [LeSt1] §8.3.14 for the Frobenius-equivariant formulation. Hecke equivariance follows since the Hecke action is by correspondences. \square

We want to bound the range of the slopes of $H_{rig}^{d-1} \left(X_{\mathbb{F}_p}^{ss}, \mathcal{E}_k^\vee(d) \right)^\vee$. To do this we will use §6.7 of [Ked2]. Since the fibre of $H_{rig}^1(A_{\mathbb{F}_p}/X_{\mathbb{F}_p})$ at a point $x \in X_{\mathbb{F}_p}$ is simply the rational Dieudonne module of $A_{\mathbb{F}_p, x}$, we note that the slopes of $H_{rig}^1(A_{\mathbb{F}_p}/X_{\mathbb{F}_p})$ lie in $[0, 1]$ (the definition is in the second paragraph of §6.7 of [Ked2]; these slopes are “pointwise slopes”). However, more importantly for us:

Proposition 28. *Assume $d \geq 2$. The slopes of $H_{rig}^1(A_{\mathbb{F}_p}/X_{\mathbb{F}_p})$ on $X_{\mathbb{F}_p}^{ss}$ are in $[\frac{1}{d}, \frac{d-1}{d}]$.*

Proof. Given a point $x \in X_{\mathbb{F}_p}^{ss}$, the corresponding abelian variety A_x is isogenous over $\overline{\mathbb{F}_p}$ to the square of a non-ordinary abelian variety A' with real multiplication by F by Proposition 5.2 of [Mil1]. The slopes of the Dieudonne module of such an abelian variety are in $[\frac{1}{d}, \frac{d-1}{d}]$ by Theorem 5.2.1 of [GoOo]. This proves the Proposition, as the Dieudonne module of A_x is $\overline{\mathbb{F}_p}$ -isogenous to the sum of two copies of the Dieudonne module of A' (remember that we are assuming $d \geq 2$). \square

Corollary 29. *The slopes of \mathcal{E}_k on $X_{\mathbb{F}_p}^{ss}$ are in $[\frac{k-2d}{d}, \frac{(k-2d)(d-1)}{d}]$.*

Using this, we are ready to prove the main result of this section. We remark first that a Tate twist by 1 decreases slopes by 1, and that dualizing sends a slope to its negative.

Theorem 30. *The slopes of $H_{rig}^{d-1} \left(X_{\mathbb{F}_p}^{ss}, \mathcal{E}_k^\vee(d) \right)^\vee$ lie in $\left[1 + \frac{k-2d}{d}, d + \frac{(k-2d)(d-1)}{d} \right]$.*

Proof. By the previous Corollary the slopes of \mathcal{E}_k on $X_{\mathbb{F}_p}^{ss}$ are in $[\frac{k-2d}{d}, \frac{(k-2d)(d-1)}{d}]$, so by the remarks before this Theorem the slopes of $\mathcal{E}_k^\vee(d)$ are in $\left[-d - \frac{(k-2d)(d-1)}{d}, -d - \frac{k-2d}{d} \right]$.

Next we apply Theorem 6.7.1 of [Ked2], a special case of which says that if S is a proper separated scheme of finite type over \mathbb{F}_p of pure dimension $d-1$ and \mathcal{F} is an overconvergent F -isocrystal on S with slopes in $[r, s]$, then the slopes of $H_{rig}^{d-1}(S, \mathcal{F})$ are in $[r, s + d - 1]$. In our situation this allows us conclude that the slopes of $H_{rig}^{d-1} \left(X_{\mathbb{F}_p}^{ss}, \mathcal{E}_k^\vee(d) \right)$ are in $\left[-d - \frac{(k-2d)(d-1)}{d}, -1 - \frac{k-2d}{d} \right]$. Dualizing we see that the slopes of $H_{rig}^{d-1} \left(X_{\mathbb{F}_p}^{ss}, \mathcal{E}_k^\vee(d) \right)^\vee$ lie in $\left[1 + \frac{k-2d}{d}, d + \frac{(k-2d)(d-1)}{d} \right]$ as desired. \square

Corollary 31. *The slopes of $H_{rig}^d \left(X_{\mathbb{F}_p}^{ord}, \mathcal{E}_k \right)$ lie in $[0, k - d]$. Thus, the part of cohomology with slopes in $\left[0, 1 + \frac{k-2d}{d} \right) \cup \left(d + \frac{(k-2d)(d-1)}{d}, k - d \right]$ lies in the image of $H_{rig}^d \left(X_{\mathbb{F}_p}, \mathcal{E}_k \right)$.*

Proof. That the slopes of $H_{rig}^d \left(X_{\mathbb{F}_p}^{ord}, \mathcal{E}_k \right)$ lie in $[0, k - d]$ follows from noting that the slopes of \mathcal{E}_k are in $[0, k - 2d]$ (since the slopes of $H_{rig}^1(A_{\mathbb{F}_p}/X_{\mathbb{F}_p})$ are in $[0, 1]$) and applying Theorem 6.7.1 of [Ked2] (not the same special case as before, but the same if you replace “proper” by “smooth”). The second part then follows by the Theorem and the excision sequence, as the part of cohomology with slopes in $\left[0, 1 + \frac{k-2d}{d} \right) \cup \left(d + \frac{(k-2d)(d-1)}{d}, k - d \right]$ necessarily gets killed when mapped to $H_{X_{\mathbb{F}_p}^{ss}, rig}^{d+1} \left(X_{\mathbb{F}_p}, \mathcal{E}_k \right)$ and hence lies in the image of $H_{rig}^d \left(X_{\mathbb{F}_p}, \mathcal{E}_k \right)$. \square

3.4. Classicity for forms of small slope, the case $d \geq 2$. Throughout this section we encourage the reader to keep part 2) of Remark 21 in mind. Recall the Frobenius correspondence Fr on X_{rig}^{ord} that we defined in §2.3, and that it overconverges. Composing Fr with U_p in one way gives the correspondence

$$r = (r_1, r_2) : X_{rig}^{ord} \rightarrow X_{rig}^{ord} \times X_{rig}^{ord}$$

with $r_1 = Fr$, $r_2 = \langle p \rangle^{-1} Fr$ (we define composition of correspondences in the opposite way to [Lau1] §1.6). As $\langle p \rangle^{\pm 1}$ commutes with the Frobenius morphism we rewrite this correspondence as the composition of $\langle p \rangle$ with the correspondence

$$r' = (r'_1, r'_2) : X_{rig}^{ord} \rightarrow X_{rig}^{ord} \times X_{rig}^{ord}$$

with $r'_1 = r'_2 = Fr$. Transferring differentials as for Fr and U_p we see that r' acts on $H^0(X_{rig}, W^\dagger(k_1, \dots, k_d, -(\sum k_i)))$ as p^{k+d} (here p^k comes from the transfer of differentials and p^d is the degree of the morphism Fr), and hence acts on $H^0(X_{rig}, W^\dagger(k_1, \dots, k_d, -(\sum k_i) + 2d))$ as p^{k-d} (by Prop. 24 and Rem. 25). Hence it acts on $H_{rig}^d(X_{\mathbb{F}_p}^{ord}, \mathcal{E}_k)$ by p^{k-d} , and therefore r acts by $\langle p \rangle p^{k-d}$. Since $H_{rig}^d(X_{\mathbb{F}_p}^{ord}, \mathcal{E}_k)$ is finite-dimensional, one-sided inverses are two-sided inverses and we can conclude that

$$(3.1) \quad Fr \circ U_p = U_p \circ Fr = \langle p \rangle p^{k-d}$$

on $H_{rig}^d(X_{\mathbb{F}_p}^{ord}, \mathcal{E}_k)$. We may conclude that the slopes of U_p acting on $H_{rig}^d(X_{\mathbb{F}_p}^{ord}, \mathcal{E}_k)$ lie in $[0, k-d]$ (as the eigenvalues of $\langle p \rangle$ are roots of unity), and furthermore we may rewrite Corollary 31 as

Lemma 32. *The part of $H_{rig}^d(X_{\mathbb{F}_p}^{ord}, \mathcal{E}_k)$ with U_p -slope in $[0, \frac{k}{d} - 2] \cup (\frac{k(d-1)}{d} + 2 - d, k - d]$ is in the image of $H_{rig}^d(X_{\mathbb{F}_p}, \mathcal{E}_k)$.*

From this, our classicality criterion follows. Let us first state the following simple consequence of Matsushima's formula:

Lemma 33. *The Hecke module $H_{rig}^d(X_{\mathbb{F}_p}, \mathcal{E}_k)$ decomposes as a direct sum of Hecke modules of K -fixed vectors associated to automorphic representations of G^* .*

Proof. The direct sum decomposition of $H_{rig}^d(X_{\mathbb{F}_p}, \mathcal{E}_k)$ reduces the question to the same assertion for the $H_{rig}^i(X_{\mathbb{F}_p}^{ord}, V^\dagger(k_1, \dots, k_d, w)^\vee)$. Since our Hecke operators are defined over \mathbb{Q} , a sequence of comparison theorems/definitions (definition of rigid cohomology, complex and rigid analytic/algebraic comparison of de Rham cohomology and flat base change) we see that the Hecke modules $H_{rig}^i(X_{\mathbb{F}_p}^{ord}, V^\dagger(k_1, \dots, k_d, w)^\vee)$ and $H_{dR}^d(X(\mathbb{C}), V(k_1, \dots, k_d, w)^\vee)$ arise as base changes of the same Hecke module over \mathbb{Q} . For the latter we have Matsushima's formula

$$H_{dR}^d(X(\mathbb{C}), V(k_1, \dots, k_d, w)^\vee) = \bigoplus_{\pi} m(\pi) \pi_f^K \otimes H^d(\mathfrak{g}^*, K_\infty; \pi_\infty \otimes \xi(k_1, \dots, k_d, w)^\vee)$$

(the standard reference is [BoWa] VII.5.2, see Thm 3.2 of [Yosh] for the formulation above and some more details) where the summation is over all irreducible admissible representations of $G^*(\mathbb{A})$, $m(\pi)$ is the multiplicity of π in the appropriate summand of $L^2(G^*(\mathbb{Q}) \backslash G^*(\mathbb{A}))$, π_f^K is the K -fixed vectors of the finite part π_f of π , $H^d(\mathfrak{g}^*, K_\infty; -)$ is $(\mathfrak{g}^*, K_\infty)$ -cohomology with trivial Hecke action and $\xi(k_1, \dots, k_d, w) = \left(\bigotimes_{i=1}^d \text{Sym}^{k_i}(Sd_i) \right) \otimes \det^{(w - \sum k_i)/2}$. As $m(\pi) \dim H^d(\mathfrak{g}^*, K_\infty; \pi_\infty \otimes \xi(k_1, \dots, k_d, w)^\vee) = 0$ unless π is the automorphic representation associated to some automorphic form of level K and weight $(k_1, \dots, k_d, -w)$, the lemma follows. \square

Theorem 34. *a) Assume that f is an overconvergent Hecke eigenform for \mathcal{H} , of weight (k_1, \dots, k_d) , character χ for the diamond operators and with U_p -slope in $[0, \frac{k}{d} - 2) \cup (\frac{k(d-1)}{d} + 2 - d, k - d]$, and assume that it is not in the image of θ . Then its system of Hecke eigenvalues for \mathcal{H} comes from the p -stabilization of a classical form of level K .*

b) Assume that f is an overconvergent Hecke eigenform for \mathcal{H} , of weight (k_1, \dots, k_d) , character χ for the diamond operators and with U_p -slope less than $\inf(k_i - 1, \frac{k}{d} - 2)$. Then its system of Hecke eigenvalues for \mathcal{H} comes from the p -stabilization of a classical form of level K .

Proof. We look here at the direct summand $\text{coker} \theta_{(k_1-2, \dots, k_d-2, k-2d)}$ of $H_{rig}^d(X_{\mathbb{F}_p}^{ord}, \mathcal{E}_k)$. By Corollary 26 part b) follows directly from a), so we may focus on a). We assume that f is not in the image of θ , hence its system of Hecke eigenvalues outside p occurs in $H_{rig}^d(X_{\mathbb{F}_p}^{ord}, \mathcal{E}_k)$, and by Lemma 32 it comes from $H_{rig}^d(X_{\mathbb{F}_p}, \mathcal{E}_k)$. Lemma 33 now gives the theorem for \mathcal{H}^p . For U_p , note that the class of f in $H_{rig}^d(X_{\mathbb{F}_p}^{ord}, \mathcal{E}_k)$ is also an eigenvector for Fr (by equation 3.1), hence for T_p as $T_p = U_p + Fr$. Since $H_{rig}^d(X_{\mathbb{F}_p}, \mathcal{E}_k) \rightarrow H_{rig}^d(X_{\mathbb{F}_p}^{ord}, \mathcal{E}_k)$ is equivariant for T_p , it follows that the T_p -eigenvalue of the class of f is the T_p -eigenvalue of the associated classical form g of level K , and that its U_p -eigenvalue satisfies the p -Hecke polynomial of g , as U_p satisfies $x^2 - T_p x + \chi(p)p^{k-d}$. Hence the U_p -eigenvalue of f agrees with that of a p -stabilization of g , which was what we wanted to prove. \square

3.5. The case $d = 1$; the Hecke modules $H_{rig}^1(X_{\mathbb{F}_p}^{ord}, V^\dagger(k-2, k-2)^\vee)$. For completeness we treat the case $d = 1$ in this section, where we can obtain better results by methods similar to those in [Col]. Let us first state the analogue of Theorem 34 that one gets by slope calculations as above, which is reminiscent of Gouvea's original conjecture for overconvergent modular forms ([Gou], Conjecture 3) :

Theorem 35. *a) Assume that f is an overconvergent Hecke eigenform for \mathcal{H} , of weight k , character χ for the diamond operators and with U_p -slope not equal to $(k-2)/2$, and assume that it is not in the image of θ . Then its system of Hecke eigenvalues for \mathcal{H} comes from the p -stabilization of a classical form of level K .*

b) Assume that f is an overconvergent Hecke eigenform for \mathcal{H} , of weight k , character χ for the diamond operators and with U_p -slope not equal to $(k-2)/2$ and less than $k-1$. Then its system of Hecke eigenvalues for \mathcal{H} comes from the p -stabilization of a classical form of level K .

We will not prove this but rather prove the following stronger theorem, which is an (weak) analogue of Corollary 7.2.1 of [Col] (see Remark 40 for a strengthening of part b)):

Theorem 36. *a) Assume that f is an overconvergent Hecke eigenform for \mathcal{H} , of weight k , character χ for the diamond operators and assume that it is not in the image of θ . Then its system of Hecke eigenvalues for \mathcal{H} is classical of level $K^p K_0(p)$.*

b) Assume that f is an overconvergent Hecke eigenform for \mathcal{H} , of weight k , character χ for the diamond operators with U_p -slope less than $k-1$. Then its system of Hecke eigenvalues is classical of level $K^p K_0(p)$.

To do this we will aim directly at the cohomology groups $H_{rig}^1(X_{\mathbb{F}_p}^{ord}, V^\dagger(k-2, k-2)^\vee)$ rather than interpreting them as summands of $H_{rig}^1\left(X_{\mathbb{F}_p}^{ord}, \text{Sym}^{k-2}(H_{rig}^1(A_{\mathbb{F}_p}/X_{\mathbb{F}_p}))\right)$. The excision sequence that we are interested in is then

$$\begin{aligned} 0 \rightarrow H_{rig}^1(X_{\mathbb{F}_p}, V^\dagger(k-2, k-2)^\vee) &\longrightarrow H_{rig}^1(X_{\mathbb{F}_p}^{ord}, V^\dagger(k-2, k-2)^\vee) \longrightarrow \\ &\longrightarrow H_{X_{\mathbb{F}_p}^{ss}, rig}^2(X_{\mathbb{F}_p}, V^\dagger(k-2, k-2)^\vee) \rightarrow H_{rig}^2(X_{\mathbb{F}_p}, V^\dagger(k-2, k-2)^\vee) \rightarrow 0 \end{aligned}$$

where the first 0 is a local H^1 which vanishes by Poincare duality (it corresponds to an H^1 on $X_{\mathbb{F}_p}^{ss}$, which is 0-dimensional), the 0 at the end comes from the fact that $X_{\mathbb{F}_p}^{ord}$ is affine and 1-dimensional so any H_{rig}^2 vanishes. Rather than slopes we will analyze this using some dimension counting entirely analogous to parts of [Col] §5 and §6. The space $H_{rig}^1(X_{\mathbb{F}_p}, V^\dagger(k-2, k-2)^\vee)$ looks (as a Hecke module) like two copies of the space of classical level K automorphic forms, by Matsushima's formula. The Hecke-equivariant quotient map

$$H^0(X_{rig}, W^\dagger(k, -k+2)) \rightarrow \text{Coker } \theta_{(k-2, k-2)} = H_{rig}^1(X_{\mathbb{F}_p}^{ord}, V^\dagger(k-2, k-2)^\vee)$$

injects the space of weight k level $K^p K_0(p)$ classical p -new forms into $H_{rig}^1(X_{\mathbb{F}_p}^{ord}, V^\dagger(k-2, k-2)^\vee)$ (this follows from Cor. 26 since these p -new forms have slope $(k-2)/2$). As they are p -new, they will not be in image of the map $H_{rig}^1(X_{\mathbb{F}_p}, V^\dagger(k-2, k-2)^\vee) \longrightarrow H_{rig}^1(X_{\mathbb{F}_p}^{ord}, V^\dagger(k-2, k-2)^\vee)$ and hence the space of weight k level $K^p K_0(p)$ classical p -new forms injects into $H_{X_{\mathbb{F}_p}^{ss}, rig}^2(X_{\mathbb{F}_p}, V^\dagger(k-2, k-2)^\vee)$.

Lemma 37. 1) Let $k \geq 3$. The space of weight k level $K^p K_0(p)$ classical p -new forms has dimension $(k-1)SS$, where SS is the number of supersingular points on $X_{\mathbb{F}_p}$.

2) The space of weight 2 level $K^p K_0(p)$ classical p -new forms has dimension $SS - 1$.

Proof. This is well known, we give a brief indication of the proof.

1) In general, one shows using the Kodaira-Spencer isomorphism and the Riemann-Roch theorem that for weight $k \geq 3$ and an arbitrary neat level K' , the space of weight k and level K' classical automorphic forms has dimension $(k-1)(g(X(K'))-1)$ where $g(X(K'))$ is the genus of the Shimura curve $X(K')$ of level K' . Let g denote the genus of X . By looking at $Y_{\mathbb{F}_p}$, one sees that the genus of Y is $2g + SS - 1$. Since the dimension of the space of weight k level $K^p K_0(p)$ classical p -old forms is twice that of the space weight k level K classical forms (each eigenform has two p -stabilizations), one gets the formula for the p -new forms.

2) Kodaira-Spencer shows that the space of weight 2 and level K' classical automorphic forms has dimension $g(X(K'))$, hence the space of weight 2 level $K^p K_0(p)$ classical p -new forms has dimension $(2g + SS - 1) - 2g = SS - 1$. \square

Lemma 38. $\dim H_{X_{\mathbb{F}_p}^{ss}, rig}^2(X_{\mathbb{F}_p}, V^\dagger(k-2, k-2)^\vee) = SS(k-1)$ for $k \geq 2$.

Proof. By Poincare duality $H_{X_{\mathbb{F}_p}^{ss}, rig}^2(X_{\mathbb{F}_p}, V^\dagger(k-2, k-2)^\vee) = H_{rig}^0(X_{\mathbb{F}_p}^{ss}, V^\dagger(k-2, k-4)^\vee)^\vee$ so since $X_{\mathbb{F}_p}^{ss}$ is (over \mathbb{C}_p , or a sufficiently large finite extension of \mathbb{Q}_p) SS points and $V^\dagger(k-2, k-4)^\vee$ has rank $k-1$, the formula follows. \square

The last ingredient of our dimension count is

Lemma 39. $H_{rig}^2(X_{\mathbb{F}_p}, V^\dagger(k-2, k-2)^\vee) = 0$ if $k \geq 3$, and the one-dimensional Hecke module corresponding to Tate twist by -1 if $k = 2$.

Proof. This follows by Matsushima's formula or other "classical methods" (e.g. degeneration of the BGG spectral sequence). \square

Adding up the dimensions in the previous lemmas we see that as Hecke modules,

$$H_{rig}^1(X_{\mathbb{F}_p}^{ord}, V^\dagger(k-2, k-2)^\vee) = H^0(Y, W(k, 2-k))$$

Here we are using that image of the injection $H_{rig}^1(X_{\mathbb{F}_p}, V^\dagger(k-2, k-2)^\vee) \rightarrow H_{rig}^1(X_{\mathbb{F}_p}^{ord}, V^\dagger(k-2, k-2)^\vee)$ is, as a Hecke module, the space of p -old forms inside $H^0(Y, W(k, 2-k))$. For \mathcal{H}^p and $\langle p \rangle$ this follows from equivariance, for U_p this uses the same trick as in the proof of Theorem 34. Thus Theorem 36 follows.

Remark 40. The fact that these two are equal as Hecke modules does not mean that the composition

$$H^0(Y, W(k, 2-k)) \hookrightarrow H^0(X_{rig}, W^\dagger(k, 2-k)) \twoheadrightarrow \text{coker } \theta_{(k-2, k-2)}$$

is an isomorphism. In the modular curve case, Coleman ([Col]) shows the equality of Hecke modules as above but also that the composition above is not an isomorphism. However, by Corollary 26, the composition is an injection, and hence an isomorphism, on slope $< k-1$ parts. This allows one to strengthen Theorem 36 b) to assert that f itself is classical (of level $K^p K_0(p)$). Corollary 7.2.1 of [Col] asserts that the analogous strengthening of a) is true in the case of modular curve. However, we cannot prove it by the same technique as we do not have q -expansions, and as a result do not know multiplicity 1 for overconvergent automorphic forms.

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